On Fourier Dimension and Salem Sets

Master's Thesis
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Contents

1 Introduction

Fourier dimension is connected to the decay of the Fourier transform of measures through energy integrals and is bounded by the Hausdorff dimension. Study of the applications of expressing energy integrals in terms of the Fourier transform and the Fourier series dates to the 1960s to works of Kahane and Salem, and Carleson. The sets with equal Fourier and Hausdorff dimensions are called Salem sets, named after the Greek mathematician Raphaël Salem who first gave a construction of such sets in 1951. Fourier transforms of measures have applications in, for example, number theory, complex analysis, and operator theory (see e.g. [10, 3]).

In Chapter 2 we go through the preliminaries including the notation, definitions, and the fundamental results used throughout this work. They concern measure theory, Fourier analysis, and probability theory, and can be found in most of the textbooks on the topics. More specific results are given as a part of the proof when required.

There are two main goals in this thesis. The first one is to introduce the Fourier dimension and to prove some of its properties. These shall be considered in Chapter 3, with comparison to the Hausdorff dimension. The second, and the bigger part of this work, is to introduce Salem sets in Chapter 4, which shall be considered through various examples of varying difficulties. These include some deterministic sets, however, emphases will be put on probabilistic examples with a focus on the images of sets and measures under some random mappings.

This thesis is mostly based on Kahane [9], Mattila [10, 11], and Ekström [2], with additions from various other sources to clarify and unify some of the parts. Efforts have been made to keep the study as self-contained as reasonably possible.

2 Preliminaries and fundamental results

We begin by going through some basic notations and results used in the text. Due to the nature of this study, some background knowledge of analysis, measure theory and mathematics in general are recommended. In this work we are going to consider mainly topics related to measure theory, Fourier analysis and probability theory, some results from number theory are also used. Due to the sheer volume of theory required we try to keep everything brief on this section. Proofs for results of great importance however are given. Sources for further reading on topics of lesser importance, from the point of view of this work, are also given. This chapter is strongly based on [10],[11],[9], [6] and [8] with the addition of some well known result from other sources.

2.1 On notation

Let X be a metric space with some metric d. The open ball of radius r about a point $x \in X$, denoted by $B(x,r)$, is the set

$$\{y \in X : d(x,y) < r\}$$

and the closed ball, $\overline{B}(x,r)$, is the set

$$\{y \in X : d(x,y) \leq r\}.$$

In case of $X = \mathbb{R}^n$, denote the unit sphere by $S^{\text{n-1}}$,

$$S^{\text{n-1}} = \{x \in \mathbb{R}^n : |x| = 1\},$$

where $|\cdot|$ denotes Euclidean norm. When we speak about a circle, we mean the space $(\mathbb{R}/2\pi) \equiv \mathbb{T}^1$. The n-dimensional torus is the product of n torus,

$$\mathbb{T}^n = \mathbb{T}^1 \times \ldots \times \mathbb{T}^1.$$

The closure of a set A is denoted by $\overline{A}$. The support of a function $f : X \to \mathbb{R}$, denoted by $\operatorname{spt} f$, is the closure of the set

$$\{x \in X : f(x) \neq 0\}.$$

If a function is continuously p-times differentiable, we say that it belongs to the space $C^p(X)$. If a function in $C^p(X)$ is also of compact support then it belongs to the space $C^p_0(X)$. A function f is said to be α-Hölder continuous if there are real constants $\alpha, C > 0$ such that

$$||f(x) - f(y)|| \leq C||x - y||^\alpha$$

for all $x, y \in X$, where $||\cdot||$ is a norm on X. We denote by $\Lambda^\alpha(X)$ the space of α-Hölder continuous functions on X. The supremum norm of a function is defined to be

$$||f||_\infty = \sup_t \{||f(t)||\}.$$

There will be some more variations on these but they are explained when encountered. Integrals over the whole space X, when defined, are denoted like

$$\int_X f(x)dx = \int f(x)dx = \int f dx.$$

In case of Lebesgue integrals, for integrable function f on $\mathbb{R}^n$, we will write

$$\int_{\mathbb{R}^n} f(x)d\mathcal{L}^{\mathrm{n}}(x) = \int f d\mathcal{L}^{\mathrm{n}} = \int f dx,$$

if this causes no confusion. The convolution of two functions f and g, when defined, is

$$(f * g)(x) = \int f(x-y)g(y)dy.$$

For $1 \leq p < \infty$, the L^p-norm of a function f, denoted by $||f||_{L^p(\mu)}$, is defined as

$$||f||_{L^p(\mu)} = (\int |f|^p d\mu)^{\frac{1}{p}}.$$

We use $L^p(\mu)$ to denote the space of μ-measurable functions whose L^p-norm are bounded. Again, in case the measure μ is the Lebesgue measure $\mathcal{L}^{\mathrm{n}}$, we simply write $L^p(\mu) = L^p$.

A function f defined on $\mathbb{R}$ is said to be concave on an interval if for all points x, y on the interval and for any $\alpha \in [0,1]$,

$$f((1-\alpha)x + \alpha y) \geq (1-\alpha)f(x) + \alpha f(y).$$

An example of a concave function is $\log(x)$ on the positive half-line. Conversely, the function is said to be convex on an interval if the above inequality is reversed. An example of a convex function is e^x.

An increasing function $\omega : [0,\infty[\to [0,\infty[$ such that $\lim_{h\to 0}\omega(h) = 0$ is called a modulus of continuity. A function f is uniformly continuous with respect to the modulus ω if

$$||f(x) - f(y)|| \leq \omega(||x-y||).$$

A function on $\mathbb{R}^n$ is said to be of Schwartz class $\mathcal{S}(\mathbb{R}^n)$ if it's of C^∞ and its partial derivative of any order tends to zero at infinity faster than $|x|^{-k}$ for

any positive integer k. Schwartz class $\mathcal{S}(\mathbb{R}^n)$ contains, for example, all the functions from C_0^∞, which are used in some of the proofs in this thesis.
For functions defined on $\mathbb{R}^n$ it's some times convenient to use the big O notation, $\mathcal{O}$. We write $f(x) = \mathcal{O}(g(x))$ if there are constants $C, C_0 > 0$ such that $|f(x)| \leq Cg(x)$ for $x \in \mathbb{R}^n$ when $|x| \geq C_0$.
If $x = a + ib$ is a complex number, we use $\mathrm{Re}(x) = a$ to denote the real part of x and $\mathrm{Im}(x) = b$ to denote the imaginary part of x.

2.2 Some measure theory

Measures are important tools used in analysis, for example, in the study of fractals and in probability theory. In this section we define and give some basic properties of measures, measurability, and related notations. Let X be a metric space with metric d.

Definition 2.1. *A set function $\mu : \{A : A \subset X\} \to [0, +\infty]$ is called a measure if the following conditions are satisfied:*

1. $\mu(\emptyset) = 0$,
2. *If $A \subset B \subset X$ then $\mu(A) \leq \mu(B)$*
3. *If $A_1, A_2, \ldots \subset X$ then $\mu(\bigcup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty \mu(A_i)$.*

We say that a set $A \subset X$ is μ-measurable if $\mu(E) = \mu(E \cap A) + \mu(E \setminus A)$ for all $E \subset X$.

A property is said to hold for almost every point $x \in X$ with respect to a measure μ, sometimes μ-a.e x or a.e x is used for abbreviation if the set of exceptional points in X is of μ measure zero. For the upcoming part about probability theory, and to define Borel sets, we give a definition for σ-algebra.

Definition 2.2. *A family Γ of subsets of X is called a σ-algebra if the following conditions are satisfied:*

1. *$\emptyset \in \Gamma$ and $X \in \Gamma$,*
2. *If $A \in \Gamma$ then $X \setminus A \in \Gamma$,*
3. *If $A_1, A_2, \ldots \in \Gamma$ then $\bigcup_{i=1}^\infty A_i \in \Gamma$.*

We call the smallest σ-algebra containing all the open sets of the metric space X Borel's σ-algebra, sometimes denoted by $\mathcal{B}(X)$. An element of Borel's σ-algebra is called a Borel set. Next, we define Borel- and Radon measures.

Definition 2.3. *Let μ be a measure on X.*

1. *A measure μ is Borel measure if every Borel set of X is μ-measurable. A Borel measure μ is Borel regular if for every set $A \subset X$ there is a Borel set $B \subset X$ such that $A \subset B$ and $\mu(A) = \mu(B)$.*

2. *A measure μ is a Radon measure if*

   (a) $\mu(K) < \infty$ for compact sets $K \subset X$

   (b) $\mu(V) = \sup\{\mu(K) : K \subset V,\ K$ is compact$\}$ for open sets $V \subset X$

   (c) $\mu(A) = \inf\{\mu(V) : A \subset V,\ V$ is open$\}$ for $A \subset X$.

For example, the Lebesgue measure $\mathcal{L}^{\mathrm{n}}$ is a Radon measure on $\mathbb{R}^n$ and measures defined via Carathéodory's construction, like the Hausdorff measure $\mathcal{H}^{\mathrm{s}}$, are Borel regular. The support of a measure μ is defined as the smallest closed set $F \subset X$ such that $\mu(X \setminus F) = 0$. Sometimes it is useful to restrict measures to sets different from their original support. Let us define the restriction of a measure.

Definition 2.4. *The restriction of a measure μ to a set $A \subset X$, denoted by $\mu_{|A}$, is defined by*

$$\mu_{|A}(E) = \mu(E \cap A), E \subset X.$$

From the definitions 2.1 and 2.4 it follows that every μ-measurable set is also $\mu_{|A}$-measurable. Also, if μ is Borel regular and $A \subset X$ with $\mu(A) < \infty$, then $\mu_{|A}$ is Borel regular [11, Theorem 1.9]. We denote the set of all Borel measures compactly supported by a set $A \subset X$ by $\mathcal{M}(A)$. The set of all Borel probability measures compactly supported by a set $A \subset X$ is denoted by $\mathcal{M}^1(A)$. Clearly, $\mathcal{M}^1(A) \subset \mathcal{M}(A)$.

Definition 2.5. *Let μ be a measure on X and let $f : X \to Y$ be a function. The image measure or push-forward of a measure μ is defined as*

$$f_*\mu(A) = \mu(f^{-1}(A)),$$

where the set $A \subset Y$.

Definition 2.6. *Let μ and ν be Radon measures on $\mathbb{R}^n$. Measure μ is absolutely continuous with respect to a measure ν, denoted by $\mu \ll \nu$, if*

$$\mu(A) = 0 \Longrightarrow \nu(A) = 0 \text{ for all } A \subset \mathbb{R}^n.$$

Next, let us define h-measures which are generalizations of the standard Hausdorff measure. These shall be used to get upper bounds for the Hausdorff dimension.

Definition 2.7. *Let $A \subset \mathbb{R}^n$, and let h be a positive valued continuous function on the positive half-line. The h-measure is defined for all $A \subset \mathbb{R}^n$ as*

$$\mathcal{H}^{\mathrm{h}}(A) = \lim_{\delta \to 0} \mathcal{H}^{\mathrm{h}}_{\delta}(A),$$

where for $0 < \delta \leq \infty$,

$$\mathcal{H}^{\mathrm{h}}_{\delta}(A) = \inf \left\{ \sum_i h(\mathrm{diam}(E_i)) : A \subset \bigcup_i E_i, \mathrm{diam}(E_i) \leq \delta \right\}.$$

The function h is called a gauge function. For example, if $0 \leq s < \infty$, the gauge function $h(x) = \alpha(s) 2^{-s} x^s$ will give the normalised Hausdorff measure $\mathcal{H}^{\mathrm{s}}$. Here $\alpha(s) = \mathcal{L}^s(B(0,1))$ for positive integer values of s, and so $\mathcal{H}^{\mathrm{s}} = \mathcal{L}^s$. For positive non-integer values of s let us use $\alpha(s) = 2^s$. Next, let us define the Hausdorff dimension of a set.

Definition 2.8. *The Hausdorff dimension of a set $A \subset \mathbb{R}^n$ is*

$$\dim_{\mathrm{H}} A = \inf\{s : \mathcal{H}^{\mathrm{s}}(A) = 0\} = \sup\{s : \mathcal{H}^{\mathrm{s}}(A) = \infty\}.$$

The following lemma tells us what happens to the Hausdorff dimension under Hölder continuous mapping. It will be used to get the upper bound for the Fourier dimension in many of the proofs we encounter. We shall later see that this direction of the estimation is generally the much easier one.

Lemma 2.9. *Let $E \subset \mathbb{R}^n$ be a compact set and f an α-Hölder continuous mapping from E to $\mathbb{R}^n$. Then*

$$\dim_{\mathrm{H}} f(E) \leq \min \left\{ n, \frac{1}{\alpha} \dim_{\mathrm{H}} E \right\}.$$

The following proof is adapted from [5, Proposition 2.3].

Proof. Let $\varepsilon > 0$. $s > \dim_{\mathrm{H}} E$ and $\{E_i\}_i$ be a δ-cover for E such that

$$\sum_i \mathrm{diam}(E_i)^s \leq \mathcal{H}^{\mathrm{s}}_{\delta}(E) + \varepsilon.$$

Then denoting by $\delta' \geq C \, \mathrm{diam}(E_i)^\alpha \geq f(\mathrm{diam}(E_i))$,

$$\begin{aligned} \mathcal{H}^{\mathrm{s}/\alpha}_{\delta'}(f(E)) &\leq \sum_i (C \, \mathrm{diam}(E_i)^\alpha)^{s/\alpha} \leq C^{s/\alpha} \sum_i \mathrm{diam}(E_i)^s \\ &\leq C^{s/\alpha}(\mathcal{H}^{\mathrm{s}}_{\delta}(E) + \varepsilon). \end{aligned}$$

Letting $\delta \to 0$, followed by $\varepsilon \to 0$, we get that

$$\mathcal{H}^{s/\alpha}(f(E)) \leq C(f, s, \alpha)\, \mathcal{H}^{s}(E) = 0. \tag{1}$$

This shows that $\dim_{\mathrm{H}} f(E) \leq s/\alpha$ and thus

$$\dim_{\mathrm{H}} f(E) \leq \frac{1}{\alpha} \dim_{\mathrm{H}} E.$$

However, in $\mathbb{R}^n$ Hausdorff dimension is bounded from above by n, so

$$\dim_{\mathrm{H}} f(E) \leq \min\left\{n, \frac{1}{\alpha} \dim_{\mathrm{H}} E\right\}.$$

□

Note, that Lemma 2.9 stays valid if E is a compact set on the circle and f is an α-Hölder continuous function from E to $\mathbb{R}^n$. Next, let us talk briefly about weak convergence or convergence in measure. We shall use it to construct measures with needed properties on many of the proofs of this study.

Definition 2.10. *The sequence $(\mu_j)_j$ of Borel measures on $\mathbb{R}^n$ converges weakly to a Borel measure μ if for all $\phi \in C_0(\mathbb{R}^n)$,*

$$\int \phi \, d\mu_j \to \int \phi \, d\mu.$$

The following theorem saves us from some trouble when constructing measures via weak convergence.

Theorem 2.11. *Any sequence $(\mu_j)_j$ of Borel measures on $\mathbb{R}^n$ such that*

$$\sup_j \mu_j(\mathbb{R}^n) < \infty$$

has a convergent subsequence.

Proof. See [11, Theorem 1.23]. □

The following theorem is called Frostman's lemma. It is used to get estimates for the Hausdorff dimension of Borel sets. Many of the results used in this study are based on it and thus we encounter it quite frequently.

Theorem 2.12. *Let $0 \leq s \leq n$. For a Borel set $A \subset \mathbb{R}^n$, $\mathcal{H}^s(A) > 0$ if and only if there is $\mu \in \mathcal{M}(A)$ such that for all $x \in \mathbb{R}^n, r > 0$,*

$$\mu(B(x,r)) \leq r^s. \tag{2}$$

In particular,

$$\dim_{\mathrm{H}} A = \sup\{s : \textit{there is } \mu \in \mathcal{M}(A) \textit{ such that (2) holds.}\}$$

Proof. We shall prove a variation of this theorem as Theorem 4.11 with use of h-measures with a positive valued increasing concave or convex function h in mind. Proof of this theorem as stated can be found in, for example, [11, Theorem 8.8]. □

Next, we define one of the most important concepts in this study, the s-energy. We shall see how it's related to the Fourier transforms, and thus, to the Fourier dimension.

Definition 2.13. *The s-energy, $s > 0$, for a Borel measure μ is*

$$I_s(\mu) = \iint |x-y|^{-s} d\mu(x) d\mu(x) = \int k_s * \mu \, d\mu,$$

where k_s is the Riesz kernel

$$k_s(x) = |x|^{-s}, \ x \in \mathbb{R}^n.$$

The following theorem gives us an example of an application of the Frostman's lemma.

Theorem 2.14. *For a Borel set $A \subset \mathbb{R}^n$,*

$$\dim_{\mathrm{H}} A = \sup\{s : \textit{there is } \mu \in \mathcal{M}(A) \textit{ such that } I_s(\mu) < \infty.\}$$

Proof. Adapted from [10, p.19-20]. We begin by noticing that if a measure μ has compact support, for $0 < t < s$, $I_s(\mu) < \infty$ implies that $I_t(\mu) < \infty$. Suppose that $\mu \in \mathcal{M}(\mathbb{R}^n)$. Also, denote the diameter of the support of measure μ by R. Then by using the equation

$$\int |x-y|^{-s} d\mu = s \int_0^\infty \frac{\mu(B(x,r))}{r^{s+1}} dr$$

and Frostman's condition (2), for $0 < t < s$,

$$I_t(\mu) \leq t \iint_0^R \frac{\mu(B(x,r))}{r^{t+1}} dr d\mu(x) \leq t\mu(\mathbb{R}^n) \int_0^R r^{s-t-1} dr < \infty.$$

By Frostman's lemma it follows that

$$\dim_{\mathrm{H}} A \leq s.$$

On the other hand, if $I_s(\mu) < \infty$, then for μ-a.e $x \in \mathbb{R}^n$

$$\int |x-y|^{-s} d\mu(x) < \infty.$$

Then there exists constant $0 < M < \infty$ such that the set

$$A = \{x : \int |x-y|^{-s} d\mu(x) < M\}$$

has positive μ-measure. Writing, for $x \in A, r > 0$,

$$\begin{aligned}(2r)^{-s}\mu_{|A}(B(x,r)) &\leq \int_{B(x,r)} r^{-s} d\mu_{|A}(y) = \int_{B(x,r)} |x-y|^{-s} d\mu_{|A}(y) \\ &\leq \int |x-y|^{-s} d\mu_{|A}(x) < M.\end{aligned}$$

Hence $\mu_{|A}(B(x,r)) \leq 2^s M r^s$ for all $x \in \mathbb{R}^n$ and $r > 0$, and by Frostman's lemma

$$\dim_{\mathrm{H}} A \geq s.$$

□

If we take that the measures in the definition of the s-energy are Borel probability measures on $\mathbb{R}^n$ with support on a compact set $A \subset \mathbb{R}^n$, and the α-energy $I_\alpha(\mu) < \infty$ for some measure $\mu \in \mathcal{M}^1(A)$, we say that the set A has positive capacity. Define the s-capacity of set A as

$$\operatorname{Cap}_s A = \sup\{I_s(\mu)^{-1}, \mu \in \mathcal{M}^1(A), \mu(\mathbb{R}^n) = 1\},$$

with respect to kernel k_s. Also, for $s > 0$ and $A \subset \mathbb{R}^n$, define the capacitary dimension as

$$\dim_{\mathrm{c}} A = \sup\{s : \operatorname{Cap}_s A > 0\} = \inf\{s : \operatorname{Cap}_s A = 0\}.$$

For $A \subset \mathbb{R}^n$, by [11, Theorem 8.7], $\dim_{\mathrm{c}} A \leq \dim_{\mathrm{H}} A$. If $A \subset \mathbb{R}^n$ is a Borel set, then, by [11, Theorem 8.9], $\dim_{\mathrm{c}} A = \dim_{\mathrm{H}} A$. Next, we consider some definitions for dimension of measures. We say that a measure μ is locally finite if for all $x \in X$ there exists $r > 0$ such that $\mu(B(x,r)) < \infty$.

Definition 2.15. *Let μ be a locally finite Borel measure on X. Then the upper- and the lower local dimension of the measure μ at point $x \in \operatorname{spt}\mu$ is, accordingly,*

$$\overline{\dim}_{\mathrm{loc}}\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

and

$$\underline{\dim}_{\mathrm{loc}}\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

If they agree on value, the local dimension of μ at point $x \in X$, $\dim_{\mathrm{loc}}\mu(x)$, is the common value.

Definition 2.16. *Let μ be a Radon measure on $\mathbb{R}^n$. The Hausdorff dimension of measure μ is*

$$\dim_{\mathrm{H}}\mu = \mu - \operatorname*{ess\,inf}_x \underline{\dim}_{\mathrm{loc}}\mu(x) = \sup\left\{s \geq 0 : \underline{\dim}_{\mathrm{loc}}\mu(x) \geq s \text{ for } \mu\text{-a.e } x\right\}.$$

The upper Hausdorff dimension of measure μ is

$$\dim_{\mathrm{H}}{}^{*}\mu = \mu - \operatorname*{ess\,sup}_x \underline{\dim}_{\mathrm{loc}}\mu(x) = \inf\left\{s \geq 0 : \underline{\dim}_{\mathrm{loc}}\mu(x) \leq s \text{ for } \mu\text{-a.e } x\right\}.$$

2.3 Notes on Fourier transforms

In this section, we give some basic properties and results on the Fourier transform. Please note that the listing given here is nowhere near exhaustive. We focus only on the results that shall be used in this study. First, let us begin with the definition.

Definition 2.17. *The Fourier transform of a function $f \in L^1$ is given by*

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int f(x)e^{-i2\pi\xi\cdot x}dx, \tag{3}$$

where $\xi \in \mathbb{R}^n$ and $(\cdot)$ denotes the Euclidean inner product,

$$x \cdot y = x_1y_1 + x_2y_2 + \ldots + x_ny_n,\ x, y \in \mathbb{R}^n.$$

We could have neglected the constant 2π. The way we have defined the Fourier transform of a function f allows us to redefine the function on a set of Lebesgue measure zero if needed. The same applies in general for all the functions in the results we give in this section. Let us give some properties of the transform.

Let $f, g \in L^1$. The product formula is

$$\int \hat{f}g dx = \int f\hat{g}dx, \tag{4}$$

and the convolution formula is, for $\xi \in \mathbb{R}^n$,

$$\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi). \tag{5}$$

Under translation with constant $a \in \mathbb{R}^n$, denote by $\tau_a(x) = x + a$, for $\xi \in \mathbb{R}^n$ we have

$$\widehat{f \circ \tau_a}(\xi) = e^{i2\pi a\cdot\xi}\hat{f}(\xi),\ \mathcal{F}(e^{i2\pi a\cdot\xi}f)(\xi) = \hat{f}(\xi - a). \tag{6}$$

Under dilation with a constant $r > 0$, denote by $\lambda_r(x) = rx$, for $\xi \in \mathbb{R}^n$ we have

$$\widehat{f \circ \lambda_r}(\xi) = r^{-n}\hat{f}(r^{-1}\xi). \tag{7}$$

Equations (4) and (5) follow from the Fubini theorem, results (6) and (7) with change of variables. For example, let us proof (4) by using Fubini theorem: For $f, g \in L^1$ we have

$$\begin{aligned}\int \hat{f}(x)g(x)dx = \iint f(t)e^{-i2\pi x\cdot t}g(x)dtdx = \int f(t)\int e^{-i2\pi x\cdot t}g(x)dxdt \\ = \int f(t)\hat{g}(t)dt.\end{aligned}$$

The following lemma is called Riemann-Lebesgue lemma. It's one of the important results in Fourier analysis and would be used to prove many of the theorems listed in this section if we didn't choose to omit the proofs.

Lemma 2.18. *Let $f \in L^1$. Then $\hat{f}$ is a continuous function and*

$$\hat{f}(\xi) \to 0,\ \text{when } |\xi| \to \infty. \tag{8}$$

Proof. See [17, Theorem 1.2]. □

Definition 2.19. *Let $f \in L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} f(t)dt = 1$, and let $\varepsilon > 0$. The function*

$$f_\varepsilon(t) = \varepsilon^{-n}f(t/\varepsilon)$$

is called an approximate identity.

Let us introduce the following property of approximate identities.

Lemma 2.20. *Let $\phi \in L^1$ be continuous at $t = 0$, and let f_ε be an approximate identity. Then*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} f_\varepsilon(t)\phi(t)dt = \phi(0).$$

The following proof is adapted from [16, Proposition 37.5].

Proof. Suppose the conditions of the theorem are satisfied. Let us do the following calculation:

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} f_\varepsilon(t)\phi(t)dt - \phi(0)\right| = \left|\int_{\mathbb{R}^n} f_\varepsilon(t)\phi(t)dt - \int_{\mathbb{R}^n} f_\varepsilon(t)\phi(0)dt\right| \\
&= \left|\int_{\mathbb{R}^n} f_\varepsilon(t)(\phi(t)-\phi(0))dt\right| \leq \int_{\mathbb{R}^n} |f_\varepsilon(t)||\phi(t)-\phi(0)|dt \\
&= \int_{|t|>\sqrt{\varepsilon}} |f_\varepsilon(t)||\phi(t)-\phi(0)|dt + \int_{|t|\leq\sqrt{\varepsilon}} |f_\varepsilon(t)||\phi(t)-\phi(0)|dt.
\end{aligned}$$

For $|t| \leq \sqrt{\varepsilon}$, we have

$$\begin{aligned}
&\int_{|t|\leq\sqrt{\varepsilon}} |f_\varepsilon(t)||\phi(t)-\phi(0)|dt \leq \sup_{|t|\leq\sqrt{\varepsilon}} |\phi(t)-\phi(0)| \int_{\mathbb{R}^n} |f_\varepsilon(t)|dt \\
&\leq \sup_{|t|\leq\sqrt{\varepsilon}} |\phi(t)-\phi(0)| \int_{\mathbb{R}^n} |f(t')|dt' = \sup_{|t|\leq\sqrt{\varepsilon}} |\phi(t)-\phi(0)| \cdot ||f||_{L^1} \to 0,
\end{aligned}$$

as $\varepsilon \to 0$, since the function ϕ is continuous, and the integral is obtained by change of variables $t' = \varepsilon t$. Similarly, for $|t| > \sqrt{\varepsilon}$, using the triangle inequality, $|\phi(t)-\phi(0)| \leq |\phi(t)| + |\phi(0)| \leq 2||\phi||_\infty$, we have

$$\begin{aligned}
\int_{|t|>\sqrt{\varepsilon}} |f_\varepsilon(t)||\phi(t)-\phi(0)|dt &\leq 2||\phi||_\infty \int_{|t|>\sqrt{\varepsilon}} |f_\varepsilon(t)|dt \\
&= 2||\phi||_\infty \int_{|t|>1/\sqrt{\varepsilon}} |f(t)|dt \to 0,
\end{aligned}$$

as $\varepsilon \to 0$, since $f(t) \to 0$ as $|t| \to \infty$, and where the integral is obtained by the change of variables $t = t\varepsilon$, hence completing the proof. □

The following formula is called the inversion formula. Again, emphasize the fact that we may need to define the function differently on a set of Lebesgue measure zero.

Theorem 2.21. *Let $f, \hat{f} \in L^1$. Then*

$$f(x) = \int \hat{f}(\xi) e^{i2\pi\xi\cdot x} d\xi. \tag{9}$$

The following proof is adapted from [10, p.27-28].

Proof. Let $\varepsilon > 0$. Define a function $\psi : \mathbb{R}^n \to \mathbb{R}$ by $\psi(x) = e^{-\pi|x|^2}$, for which $\int_{\mathbb{R}^n} \psi(x)dx = 1$ (see [18, Appendix 2]). Now by definition, the function

$$\psi_\varepsilon(x) = \varepsilon^{-n} e^{-\pi|x/\varepsilon|^2}$$

is an approximate identity. Also, we have $\hat{\psi} = \psi$, which can be seen from the following calculation: Completing the square

$$\begin{aligned}
&-\pi|x|^2 - i2\pi x\cdot\xi \\
&= -\pi\sum_{j=1}^{n}(x_j\cdot x_j)^2 - \pi\sum_{j=1}^{n}(x_j\cdot\xi_j) - \pi\sum_{j=1}^{n} i^2(\xi_j\cdot\xi_j)^2 + \pi\sum_{j=1}^{n} i^2(\xi_j\cdot\xi_j)^2 \\
&= -\pi\sum_{j=1}^{n}(x_j + i\xi_j)^2 - \pi|\xi|^2,
\end{aligned}$$

we can write

$$\begin{aligned}
\hat{\psi}(\xi) = \int_{\mathbb{R}^n} e^{-\pi|x|^2} e^{-i2\pi x\cdot\xi}dx &= e^{-\pi|\xi|^2}\int_{\mathbb{R}^n} e^{-\pi\sum_{j=1}^{n}(x_j+i\xi_j)^2}dx \\
&= e^{-\pi|\xi|^2}\int_{\mathbb{R}} e^{-\pi(x_1+i\xi_1)^2}dx \times \ldots \times \int_{\mathbb{R}} e^{-\pi(x_n+i\xi_n)^2}dx \\
&= e^{-\pi|\xi|^2}\left(\int_{\mathbb{R}} e^{-\pi y^2}dy\right)^n = e^{-\pi|\xi|^2}.
\end{aligned}$$

Above we have changed variables $y = x_j + i\xi_j$ for each $j = 1, \ldots, n$, which is justified by noting that for any $s \in \mathbb{R}$

$$\int_{-\infty}^{\infty}\frac{d}{ds}\left(e^{-\pi(x+is)^2}\right)dx = i\int_{-\infty}^{\infty}\frac{d}{dx}\left(e^{-\pi(x+is)^2}\right)dx = 0,$$

implying that the value of the integral is independent of s. Hence by (7) we have

$$\widehat{\psi\circ\lambda_\varepsilon}(\xi) = \varepsilon^{-n}e^{-\pi|\xi/\varepsilon|^2} = \hat{\psi}_\varepsilon(\xi).$$

If we denote by

$$I_\varepsilon(\xi) = \int \hat{f}(x)e^{-\pi\varepsilon^2|x|^2}e^{i2\pi\xi\cdot x}dx, \tag{10}$$

then by Lebesgue's dominated convergence theorem

$$I_\varepsilon \to \int \hat{f}(x)e^{i2\pi\xi\cdot x}dx, \text{ as } \varepsilon\to 0.$$

Next, writing

$$g_x(y) = e^{-\pi\varepsilon^2|y|^2}e^{i2\pi x\cdot y},$$

we have by the translation property (6) that

$$\hat{g}_x(y) = \hat{\psi}_\varepsilon(y-x) = \varepsilon^{-n}\psi((x-y)/\varepsilon).$$

Applying the product formula (4) to equation (10) we have

$$I_\varepsilon = \int \hat{f} g_x = \int f \hat{g}_x = \psi_\varepsilon * f(x) \to f(x),$$

as $\varepsilon \to 0$ Lebesgue-almost everywhere by Lemma 2.20, which proves the formula. $\square$

For functions $f, g \in L^2$ we have

$$\int f\overline{g} = \int \hat{f}\overline{\hat{g}} \quad \text{Parseval} \tag{11}$$

$$||f||_2 = ||\hat{f}||_2 \quad \text{Plancherel} \tag{12}$$

Equation (11) follows from the inversion formula (9) by

$$\int f(x)\overline{g}(x)dx = \int \hat{\hat{f}}(-x)\overline{g}(x)dx = \int \hat{\hat{f}}(x)\overline{g}(-x)dx = \int \hat{f}(x)\overline{\hat{g}}(x)dx,$$

where last equality is due to (4) and the definition of Fourier transform. Equation (12) follows from (11) by choosing function $g(x) = f(x)$, since

$$|f(x)|^2 = f(x) \cdot \overline{f}(x).$$

Next, some result on the Fourier transform of measures. Let us begin with the definition.

Definition 2.22. *The Fourier transform of a finite Borel measure μ is*

$$\hat{\mu}(\xi) = \int e^{-i2\pi\xi\cdot x} d\mu(x), \tag{13}$$

where $\xi \in \mathbb{R}^n$.

Let $f \in L^1$ and $\mu, \nu \in \mathcal{M}(\mathbb{R}^n)$. Then we have the product formula

$$\int \hat{\mu} f dx = \int \hat{f} d\mu, \tag{14}$$

and the convolution formula, for $\xi \in \mathbb{R}^n$,

$$\widehat{(f * \mu)}(\xi) = \hat{f}(\xi)\hat{\mu}(\xi). \tag{15}$$

Also, we have the product formula for the Fourier transform of measures

$$\int \hat{\mu} d\nu = \int \hat{\nu} d\mu, \tag{16}$$

and the convolution formula for the Fourier transform of measures, for $\xi \in \mathbb{R}^n$,

$$\widehat{(\mu * \nu)}(\xi) = \hat{\mu}(\xi)\hat{\nu}(\xi). \tag{17}$$

Here the convolution of a function $f \in L^1$ and measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ is

$$(f * \mu)(x) = \int f(x-y)d\mu(y).$$

Once again, properties (14),(15) and (16) follow from the Fubini theorem. Let us prove (17):

Proof. Let $\mu, \nu \in \mathcal{M}(\mathbb{R}^n)$. Then by definition

$$\begin{aligned}\widehat{(\nu * \mu)}(\xi) &= \int e^{-i2\pi\xi\cdot x} d\nu * \mu(x) = \iint e^{-i2\pi\xi\cdot(y+x)} d\nu(y)d\mu(x) \\ &= \int e^{-i2\pi\xi\cdot x} \int e^{-i2\pi\xi\cdot y} d\nu(y)d\mu(x) = \hat{\nu}(\xi) \int e^{-i2\pi\xi\cdot x} d\mu(x) \\ &= \hat{\nu}(\xi)\hat{\mu}(\xi),\end{aligned}$$

as wanted. □

Next, we consider some results for measures and functions supported by the unit cube. These results are also used for functions defined on torus $\mathbb{T}^n$ and can be applied for 1-periodic functions of $\mathbb{R}^n$, given that the conditions under which the results hold are satisfied, see [10, Chapter 3.10] and [6, Chapter 3].

Definition 2.23. *Let $Q_n = \{x \in \mathbb{R}^n : 0 \leq x_j \leq 1, j = 1, \ldots, n\}$ be the unit cube. Fourier coefficients of a measure $\mu \in \mathcal{M}(Q_n)$ are given by*

$$\hat{\mu}(z) = \int_{Q_n} e^{-i2\pi z\cdot x} d\mu(x), z \in \mathbb{Z}^n.$$

The inversion formula (9) becomes as follows.

Theorem 2.24. *Let $f \in L^1(Q_n)$ such that $\sum_{z\in\mathbb{Z}^n} |\hat{f}(z)| < \infty$. Then f is continuous and*

$$f(x) = \sum_{z\in\mathbb{Z}^n} \hat{f}(z)e^{i2\pi z\cdot x}, x \in Q_n.$$

Proof. See [6, Proposition 3.1.14] □

Once again, the identity in Theorem 2.24 is understood to hold for almost every point $x \in Q_n$. The following theorem is Parseval's equation for Q_n.

Theorem 2.25. *Let $f, g \in L^2(Q_n)$. Then*

$$\sum_{z\in\mathbb{Z}^n} \hat{f}(z)\overline{\hat{g}(z)} = \int_{Q_n} f(x)\overline{g(x)}dx.$$

Proof. See [6, Proposition 3.1.16] □

The following is Theorem 2.25 in case of signed measure, that is, not necessarily positive set function satisfying conditions 1), and equality on 3) for disjoint sets $\{A_i\}_{i\in\mathbb{N}}$, of the Definition 2.1.

Theorem 2.26. *Let f be a continuous function on Q_n and μ a signed measure. Then*

$$\sum_{z\in\mathbb{Z}^n} \hat{f}(z)\overline{\hat{\mu}(z)} = \int_{Q_n} f(x)d\mu(x),$$

if $\sum_{z\in\mathbb{Z}^n} \hat{f}(z)\overline{\hat{\mu}(z)}$ converges.

Proof. Let f be a continuous function of the unit cube Q_n with the Fourier series

$$f(x) = \sum_{z\in\mathbb{Z}^n} \hat{f}(z)e^{i2\pi z\cdot x}.$$

Then by Fubini theorem

$$\begin{aligned}\int_{Q_n} f(x)d\mu(x) = \int_{Q_n} \sum_{z\in\mathbb{Z}^n} \hat{f}(z)e^{i2\pi z\cdot x}d\mu(x) = \sum_{z\in\mathbb{Z}^n} \hat{f}(z) \int_{Q_n} e^{i2\pi z\cdot x}d\mu(x) \\ = \sum_{z\in\mathbb{Z}^n} \hat{f}(z)\overline{\hat{\mu}(z)},\end{aligned}$$

where the change in the order of integration and summation is justified given that the series $\sum_{z\in\mathbb{Z}^n} \hat{f}(z)\overline{\hat{\mu}(z)}$ converges. □

For calculation of Fourier transforms of radial functions let us define Bessel functions of the first kind. For our use case, the definition is given in the Poisson representation formula.

Definition 2.27. *Let $m > -\frac{1}{2}$. A function $J_m : [0, \infty) \to \mathbb{R}$,*

$$J_m(u) = \frac{(u/2)^m}{\Gamma(m+1/2)\Gamma(1/2)} \int_{-1}^{1} e^{iut}(1-t^2)^{m-\frac{1}{2}}dt \tag{18}$$

is called the Bessel function of m:th order, where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is the gamma function.

Let us list some of the properties of the Bessel functions. First, some estimates

1. $|J_m(t)| \leq C(m)t^m$, $t > 0$
2. $|J_m(t)| \leq c(m)t^{-\frac{1}{2}}$, $t > 0$,

followed by the recursion formulas

i) $\frac{d}{dt}(t^{-m}J_m(t)) = -t^m J_{m+1}(t)$,

ii) $\frac{d}{dt}(t^m J_m(t)) = t^m J_{m-1}(t)$.

Proof of properties 1.), 2.), i) and ii) can be found in [6, Appendix B].

Theorem 2.28. *The Fourier transform of a radial function $f \in L^1(\mathbb{R}^n)$, $f(x) = \psi(|x|)$, is given by*

$$\hat{f}(t) = c(n)|t|^{-(n-2)/2} \int_0^\infty \psi(s) J_{(n-2)/2}(2\pi|t|s)s^{n/2} ds,$$

where $n \geq 2$.

Since we indirectly base most of our results on this fact, let us prove the formula the long way. The following proof is adapted from [6, Appendix B, Appendix D].

Proof. Let $f \in L^1$, $f(t) = \psi(|t|)$ be a radial function defined on $\mathbb{R}^n$, where ψ is a function $\psi : [0, \infty[\to \mathbb{R}^n$. Then

$$\begin{aligned} \hat{f}(u) &= \int f(t) e^{-i2\pi u \cdot t} dt \\ &= \int_0^\infty \int_{S^{n-1}} \psi(r) e^{-i2\pi u \cdot r\theta} d\theta r^{n-1} dr \\ &= \int_0^\infty \psi(r) \hat{\sigma}(ru) r^{n-1} dr, \end{aligned} \tag{19}$$

where σ is the surface measure of S^{n-1}. Let us calculate the value of the Fourier transform $\hat{\sigma}$. We do this by changing the Cartesian coordinates to spherical coordinates as follows. For $t \in \mathbb{R}^n$, $|t| = R \geq 0$, write

$$\int_{RS^{n-1}} f(t) d\sigma(t) = \int_0^\pi \int_0^\pi \cdots \int_0^{2\pi} f(t(\phi)) J(n, R, \phi) d\phi_{n-1} \dots d\phi_1, \tag{20}$$

where

$$\begin{cases} t_1 & = R\cos\phi_1 \\ t_2 & = R\sin\phi_1\cos\phi_2 \\ t_3 & = R\sin\phi_1\sin\phi_2\cos\phi_3 \\ \vdots & \\ t_{n-1} & = R\sin\phi_1\sin\phi_2\ldots\sin\phi_{n-2}\cos\phi_{n-1} \\ t_n & = R\sin\phi_1\sin\phi_2\ldots\sin\phi_{n-2}\sin\phi_{n-1}. \end{cases}$$

Variables given above are $0 \leq \phi_1 \leq \ldots \leq \phi_{n-2} \leq \pi$ corresponding to the angle between t and the 'zenith' direction, and $0 \leq \phi_{n-1} < 2\pi$ corresponding to 'azimuth' angle of the three-dimensional spherical coordinates. Also, $t(\phi) = (t_1(\phi_1,\ldots,\phi_{n-1}),\ldots,t_n(\phi_1,\ldots,\phi_{n-1}))$ and finally

$$J(n,R,\phi) = R^{n-1}\sin^{n-2}\phi_1\sin^{n-3}\phi_2\ldots\sin^2\phi_{n-3}\sin\phi_{n-2}$$

is the Jacobian. Next, we would like to express (20) as an iterated integral. For that, let $\phi' = (\phi_2,\ldots,\phi_{n-1})$ and write

$$t' = t'(\phi') = (\cos\phi_2, \sin\phi_2\cos\phi_3,\ldots,\sin\phi_2\cdots\sin\phi_{n-2}\sin\phi_{n-1}). \tag{21}$$

Then writing (20) with the new notation (21), we have

$$\begin{aligned} &\int_{RS^{n-1}} f(t)d\sigma(t) \\ &= \int_0^\pi\int_0^\pi\cdots\int_0^{2\pi} f(R\cos\phi_1, R\sin(\phi_1)t'(\phi'))J(n-1,1,\phi')d\phi'\frac{Rd\phi_1}{(R\sin\phi_1)^{2-n}} \\ &= \int_0^\pi\int_{S^{n-2}} f(R\cos\phi_1, R\sin(\phi_1)t'(\phi'))d\sigma(t')\frac{Rd\phi_1}{(R\sin\phi_1)^{2-n}}. \end{aligned} \tag{22}$$

Denoting by $s = R\cos\phi_1$, $0<\phi_1<1$, we have

$$ds = -R\sin\phi_1 d\phi_1, \sqrt{R^2-s^2} = R\sin\phi_1, \tag{23}$$

and applying (23), we can write (22) as

$$\begin{aligned} &\int_{-R}^R\left[\int_{S^{n-2}} f(s,\sqrt{R^2-s^2}\theta)d\theta\right](\sqrt{R^2-s^2})^{n-2}\frac{Rds}{\sqrt{R^2-s^2}} \\ &= \int_{-R}^R\left[\int_{\sqrt{R^2-s^2}S^{n-2}} f(s,\theta)d\theta\right]\frac{Rds}{\sqrt{R^2-s^2}}. \end{aligned} \tag{24}$$

Now, for $t \in \mathbb{R}^n \setminus \{0\}$, denote by $\tilde{t} = t/|t|$ and let A be a matrix from the orthogonal group of $\mathbb{R}^n$ such that

$$Ae_1 = \tilde{t}, \tag{25}$$

where $e_1 = (1, 0, \ldots, 0)$. In the following, we use the fact that the definition of orthogonality implies $A^T = A^{-1}$, and since the orthogonal group consists of reflections and rotations, by using (25), and (24) when $R = 1$, for function ϕ defined on the real line we have

$$\int_{S^{n-1}} \phi(t \cdot \theta) d\theta = \int_{S^{n-1}} \phi(|t|(\tilde{t} \cdot \theta)) d\theta = \int_{S^{n-1}} \phi(|t|(Ae_1 \cdot \theta)) d\theta$$
$$= \int_{S^{n-1}} \phi(|t|(e_1 \cdot A^{-1}\theta)) d\theta = \int_{S^{n-1}} \phi(|t|\theta_1) d\theta$$
$$= \int_{-1}^{1} \phi(|t|s) c_{n-2} (\sqrt{1-s^2})^{n-2} \frac{ds}{\sqrt{1-s^2}} = c_{n-2} \int_{-1}^{1} \phi(|t|s)(\sqrt{1-s^2})^{n-3} ds, \tag{26}$$

where $c_{n-2} = 2\pi^{\frac{n-1}{2}} \Gamma(\frac{n-1}{2})^{-1}$ is the surface area of S^{n-2}. Now it's a straightforward calculation to apply (26) and (18) to obtain

$$\hat{\sigma}(u) = \int_{S^{n-1}} e^{-i2\pi u \cdot t} dt$$
$$= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_{-1}^{1} e^{i2\pi |u| s} (1-s^2)^{\frac{n-2}{2}} \frac{ds}{\sqrt{1-s^2}}$$
$$= \frac{2\pi^{\frac{n-1}{2}} \Gamma(\frac{n-1}{2} + \frac{1}{2}) \Gamma(\frac{1}{2})}{(\pi |u|)^{\frac{n-2}{2}} \Gamma(\frac{n-1}{2})} J_{\frac{n-2}{2}}(2\pi |u|) = \frac{2\pi}{|u|^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(2\pi |u|). \tag{27}$$

Thus applying (27) to (19) we have

$$\hat{f}(u) = 2\pi |u|^{-\frac{n-2}{2}} \int_0^\infty \psi(s) J_{\frac{n-2}{2}}(2\pi s|u|) s^{\frac{n}{2}} ds,$$

as wanted. □

Example 2.29. *Let us calculate the Fourier transform of the characteristic function of the unit ball of* $\mathbb{R}^n, n \geq 2$*: Let* $f(x) = \chi_{B(0,1)}(x)$*. Now the function* f *is radial, so the Fourier transform is given by*

$$\hat{f}(t) = c(n)|t|^{-(n-2)/2} \int_0^\infty \chi_{B(0,1)}(s) J_{(n-2)/2}(2\pi |t| s) s^{n/2} ds.$$

Changing the variables, $s := |t|s$*, gives us*

$$\hat{f}(t) = c|t|^{-n} \int_0^{|t|^{-1}} J_{(n-2)/2}(2\pi s) s^{n/2} ds.$$

Estimating by using property 1.) of Bessel functions

$$|\hat{f}(t)| \leq c|t|^{-n} \int_0^{|t|^{-1}} \left|J_{(n-2)/2}(2\pi s)s^{n/2}\right| ds \leq c|t|^{-n} \int_0^{|t|^{-1}} C(n)s^{-\frac{1}{2}}s^{n/2}ds$$
$$\leq C|t|^{-n} \int_0^{|t|^{-1}} s^{(n-1)/2}ds = C(n)|t|^{-(n-1)/2},$$

where $C(n)$ is a constant. Thus for all $t \in \mathbb{R}^n$,

$$|\widehat{\chi_{B(0,1)}}(t)| \leq C(n)|t|^{-(n-1)/2}.$$

Theorem 2.30. *For $0 < s < n$ there is a positive and finite constant $\gamma(n, s)$ such that for $\phi \in \mathcal{S}(\mathbb{R}^n)$*

$$\int k_s\hat{\phi} = \gamma(n, s) \int k_{n-s}\phi.$$

Proof. See [10, Theorem 3.6]. □

The above theorem says that the Fourier transform of k_s exists in a distributional sense, as so-called tempered distribution. More in-depth consideration of tempered distributions can be found in [6]. The following theorem will be the motivating fact behind the definition of the Fourier dimension.

Theorem 2.31. *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $0 < s < n$. Then*

$$I_s(\mu) = \gamma(n, s) \int |\hat{\mu}(x)|^2|x|^{s-n}dx, \tag{28}$$

where $\gamma(n, s) = \pi^{s-n/2}\frac{\Gamma((n-s)/2)}{\Gamma(s/2)}$ is a constant.

The following proof is adapted from [10, Theorem 3.10].

Proof. Formally, by Parseval and convolution formulas, and applying Theorem 2.30,

$$I_s(\mu) = \int k_s * \mu d\mu = \int \widehat{(k_s * \mu)}\overline{\hat{\mu}} = \int \hat{k}_s|\hat{\mu}|^2 = \gamma(n, s) \int |\hat{\mu}(x)|^2|x|^{s-n}dx.$$

However, since $\hat{k}_s$ exists only in a distributional sense, let us show that the use of Parseval and convolution formulas is justified. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be real valued. Also denote by $\phi_-(x) = \phi(-x)$ Then by change of variables

$$I_s(\phi) = \iint k_s(y - x)\phi(x)\phi(y)dxdy$$
$$= \iint k_s(z)\phi(y - z)\phi(y)dzdy = \int k_s(x)(\phi_- * \phi)(x)dx.$$

Now, by applying convolution formula we have

$$(\phi_- * \phi)(x) = (\hat{\hat{\phi}} * \phi)(x) = \hat{\phi}(x)\overline{\hat{\phi}}(x) = |\hat{\phi}(x)|^2,$$

so by Theorem 2.30

$$I_s(\phi) = \gamma(n,s) \int k_{n-s}(x)|\hat{\phi}(x)|^2 dx = \gamma(n,s) \int |x|^{s-n}|\hat{\phi}(x)|^2 dx,$$

which proves it for a measure defined as $d\mu = \phi dx$. For general measure μ we make the approximation $\mu_\varepsilon(x) = (\psi_\varepsilon * \mu)(x)$, $x \in \mathbb{R}^n$, where $\psi \in C_0^\infty$ is a positive function such that $\int \psi dx = 1$. Applying the above calculation for the function $\phi = \mu_\varepsilon$ we have by Fubini theorem

$$\begin{aligned}
&\iint \left(\iint |x-y|^{-s} \psi_\varepsilon(x-z)\psi_\varepsilon(y-w) dx dy \right) d\mu(z) d\mu(w) \\
&= \iint \left(|x-y|^{-s} \int \psi_\varepsilon(x-z) d\mu(z) \int \psi_\varepsilon(y-w) d\mu(w) \right) dx dy \\
&= I_s(\mu_\varepsilon) = \gamma(n,s) \int |x|^{s-n} |\hat{\mu}(x)|^2 |\hat{\psi}(\varepsilon x)|^2 dx \\
&\longrightarrow \gamma(n,s) \int |x|^{s-n}|\hat{\mu}(x)|^2 dx, \text{ as } \varepsilon \to 0.
\end{aligned}$$

Changing the variables $u = (x-z)/\varepsilon$ and $v = (y-w)/\varepsilon$, the inner integral of the first term on the above calculation becomes

$$\begin{aligned}
&\iint |x-y|^{-s} \psi_\varepsilon(x-z)\psi_\varepsilon(y-w) dx dy \\
&= \iint |(u-v)\varepsilon + z - w|^{-s} \psi_\varepsilon(u)\psi_\varepsilon(v) du dv,
\end{aligned}$$

which tends to $|z-w|^{-s}$ as $\varepsilon \to 0$ and $z \neq w$. Using the estimate

$$\iint |x-y|^{-s} \psi_\varepsilon(x-z)\psi_\varepsilon(y-w) dx dy \leq C|z-w|^{-s},$$

where $C > 0$ is a constant, we get the theorem if $I_s(\mu) < \infty$. If $I_s(\mu) = \infty$, by Fatou's lemma

$$\begin{aligned}
\infty &\leq \liminf_{\varepsilon \to 0} \iint \left(\iint |x-y|^{-s} \psi_\varepsilon(x-z)\psi_\varepsilon(y-w) dx dy \right) d\mu(z) d\mu(w) \\
&= \gamma(n,s) \liminf_{\varepsilon \to 0} \int |x|^{s-n} |\hat{\mu}(x)|^2 |\hat{\psi}(\varepsilon x)|^2 dx \\
&= \gamma(n,s) \int |x|^{s-n} |\hat{\mu}(x)|^2 dx,
\end{aligned}$$

which proves it. □

2.4 Some probability theory

In this section, we talk briefly about some basic definitions, terminology, and results on probability. These shall be used when we begin the construction of Salem sets that are probabilistic, in other words, random. It may be more pleasant to skip this section for now and return before moving to Chapter 4. Mainly all of the results we need concern expectation value.
Let us begin with the introduction of probability space $(X, \Gamma, \mathcal{P})$. Here X denotes the sample space, Γ is a σ-algebra on X and $\mathcal{P}$ is a probability measure on (X, Γ). We require that the axioms of probability are satisfied. These are

1. $\mathcal{P}(X) = 1$,
2. if $A \in \Gamma$, it follows that $\mathcal{P}(A) \in [0, 1]$, and
3. $\mathcal{P}(\bigcup A_i) = \sum \mathcal{P}(A_i)$ for any countable disjoint collection $\{A_i\}_i \in \Gamma$.

Also, we require that the measure $\mathcal{P}$ is complete, in other words, that every subset of a set of measure zero is also measurable; If $A \subset B \in \Gamma$ with $\mathcal{P}(B) = 0$, then $\mathcal{P}(A) = 0$. With these assumptions fulfilled, the triplet $(X, \Gamma, \mathcal{P})$ is called a probability space. An element of Γ is called an event. If we only talk about elements of some space X, we always assume that the probability space is defined. It is not generally required that the probability space is complete but the reason why we choose to do so becomes clear shortly. We shall encounter results that hold almost surely, a.s, for abbreviation. What we mean by that is, if A holds a.s, then $\mathcal{P}(A) = 1$.

Definition 2.32. *A collection of events $\{A_n\}_{n=1}^N$, $N \in \mathbb{N}$, of probability space $(X, \Gamma, \mathcal{P})$ is called independent if for all index combinations $\{n_1, \dots, n_k\} \subset \{1, \dots, N\}$, where $k \in \{2, 3, \dots, N\}$, we have*

$$\mathcal{P}\left(A_{n_1} \cap \dots \cap A_{n_k}\right) = \mathcal{P}(A_{n_1}) \cdots \mathcal{P}(A_{n_k}).$$

Next, let us talk briefly about random variables. In our case, a random variable is a $\mathcal{P}$-measurable function $Y : X \to \mathbb{R}$. This choice also becomes evident shortly. When it comes to probability, our main tool will be the expectation value $\mathbb{E}$:

Definition 2.33. *Let $(X, \Gamma, \mathcal{P})$ be a probability space. If $Y \in L^1(X)$, with respect to $\mathcal{P}$, is a random variable, the expectation value of Y is defined as*

$$\mathbb{E}(Y) = \int_X Y(\omega) d\mathcal{P}(\omega).$$

For every real valued random variable Y, and real valued u, we define the characteristic function

$$M_Y(u) = \mathbb{E}(e^{iuY}) = \int_X e^{iu\omega} d\mathcal{P}, \tag{29}$$

which always exists since mapping $x \to e^{iuY}$ is continuous and

$$|M_Y(u)| \leq \int_X 1 d\mathcal{P} = \mathcal{P}(X) = 1 < \infty.$$

Generally, random variables obey some distribution. In our case that will be the Gaussian normal distribution with probability density function

$$f(t, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}}, \tag{30}$$

where μ is the expectation value of the distribution and σ^2 is the variance of the distribution. We say that a random variable ξ is Gaussian if for $-\infty < \lambda < \infty$,

$$\mathbb{E}(e^{\lambda\xi}) \leq e^{\frac{\lambda^2}{2}}. \tag{31}$$

In some sources, Gaussian random variables are sometimes also called Gaussian normal variables, or subnormal variables. A sequence of independent Gaussian variables $(\xi_n)_n$ is called a subnormal sequence. Let us use the value $\mu = 0$. Apart from reducing the number of constants in the upcoming calculations, it's also useful when considering Gaussian Hilbert spaces [8, p.4]. Depending on the case in hand, we use the value of σ^2 which reduces the largest amount of constants.
For a Gaussian random variable ξ we have the characteristic function

$$M_\xi(u) = \hat{f}(iu) = e^{\mu u} e^{\frac{1}{2}\sigma^2 u^2}, \tag{32}$$

where the function f is given by (30). The function defined on (32) is well defined; the Fourier transform of function f is an integral of the probability density function, which is bounded and continuous, and the integral is taken over a probability space whose measure is finite.

For example, if we take $\sigma^2 = 1$. Then $(\mathbb{R}^n, \overline{\mathcal{B}(\mathbb{R}^n)}, \mu)$ is a probability space, where $\overline{\mathcal{B}(\mathbb{R}^n)}$ is the completion of $\mathcal{B}(\mathbb{R}^n)$ and $d\mu = (2\pi)^{-n/2} e^{-|x|^2/2} dx$, where dx denotes Lebesgue measure. As en extra note, we could have taken the above setting as our choice of probability space and the properties we speak as 'Gaussian' would follow from the choice of measure without any mention about distributions.

Next, let us give some basic properties of the expectation value. Let χ_A be the characteristic function of event A. Then

$$\mathbb{E}\left(\chi_A(x)\right) = \int_X \chi_A(\omega) d\mathcal{P}(\omega) = \int_A d\mathcal{P}(x) = \mathcal{P}(A).$$

If $X_1, X_2, \ldots$ is a finite or countable sequence of independent real random variables such that $X_j \in L^1(X)$ for all j and $\prod_n |X_n| \in L^1(X)$ then

$$\mathbb{E}\left(\prod_n X_n\right) = \prod_n \mathbb{E}(X_n).$$

The following lemma is known as Borel-Cantelli lemma. It's an example of so-called zero-one law, that is, given some conditions an event must have probability zero or one. Plenty of these exist for different settings, but we are only concerned about this one to prove Theorem 4.6.

Lemma 2.34. *If* $\sum_{n=1}^{\infty} \mathcal{P}(A_n) < \infty$, *then*

$$\mathcal{P}(\limsup_{n\to\infty} A_n) = 0.$$

If $A_1, A_2, \ldots$ *are independent and if* $\sum_{n=1}^{\infty} \mathcal{P}(A_n) = \infty$, *then*

$$\mathcal{P}(\limsup_{n\to\infty} A_n) = 1.$$

Here the limit superior of sequence of events $(A_n)_n$ is defined as

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{j>n}^{\infty} A_j.$$

Proof. Adapted from [15, Theorem 46]. For the first part, let $\{A_n\}_{n\in\mathbb{N}}$ be a family of events. Since we have

$$\bigcup_{n=1}^{\infty} A_n \supseteq \bigcup_{n=2}^{\infty} A_n \supseteq \ldots \supseteq \bigcup_{n=k}^{\infty} A_n \supseteq \ldots \tag{33}$$

it follows from the definition of limit superior that

$$\mathcal{P}(\limsup_{n\to\infty} A_n) = \lim_{k\to\infty} \mathcal{P}\left(\bigcup_{n=k}^{\infty} A_n\right) \leq \lim_{k\to\infty} \sum_{n=k}^{\infty} \mathcal{P}(A_n).$$

By assumption, $\sum_{n=1}^{\infty} \mathcal{P}(A_n) < \infty$, so we have $\lim_{k\to\infty} \sum_{n=k}^{\infty} \mathcal{P}(A_n) = 0$, which finishes the first part.

For the second part, let $\{A_n\}_{n\in\mathbb{N}}$ be a family of independent events. Since the events A_n are independent, using (33) for the complement of event A_n we have

$$\mathcal{P}(\limsup_{n\to\infty} A_n^c) = \lim_{k\to\infty} \mathcal{P}\left(\bigcap_{n=k}^{\infty} A_n^c\right) = \lim_{k\to\infty} \prod_{n=k}^{\infty} \mathcal{P}(A_n^c) = \lim_{k\to\infty} \prod_{n=k}^{\infty} (1 - \mathcal{P}(A_n)).$$

Using the fact that $1 - x \leq e^{-x}$ for all $x \in \mathbb{R}$, and then the assumption $\sum_{n=1}^{\infty} \mathcal{P}(A_n) = \infty$, we have

$$\lim_{k\to\infty} \prod_{n=k}^{\infty} (1 - \mathcal{P}(A_n)) \leq \lim_{k\to\infty} e^{-\sum_{n=k}^{\infty} \mathcal{P}(A_n)} = 0.$$

Thus $\mathcal{P}(\limsup_{n\to\infty} A_n) = 1 - \mathcal{P}(\limsup_{n\to\infty} A_n^c) = 1.$ □

Gaussian process and fractional Brownian motion

Next we move to a little more technical topic that shall be considered later in Section 4.2.3. For our needs, let us skip the background part about stochastic processes which can be found in, for example, [8, Appendix B].
We say that a stochastic process (X_t) indexed by $\mathbb{R}^n$, i.e $t \in \mathbb{R}^n$, with values in $\mathbb{R}^d$ is Gaussian, if

$$\mathbb{E}\left(|X_t - X_s|^2\right) = d|t - s|^{\gamma},\ t, s \in \mathbb{R}^n,$$

where $(|\cdot|)$ is the Euclidean norm. The process exists for values $0 \leq \gamma \leq 2$ and we call it (n, d, γ)-process. The case when $n = \gamma = 1$ is called the Wiener process, or Brownian motion, and we shall give it a more in-depth consideration in Section 4.2.1. A process is called continuous-time if the index takes values from a continuous set of values. The higher dimensional version $(n \geq 2)$ of a (n, d, γ)-process is called fractional Brownian motion, if it's of continuos-time. Later on this section, we are going to give a brief construction of the (n, d, γ)-process and state the Dudley-Fernique theorem which tells that an a.s continuous version, $X(t)$, of the process exists for $\beta = \gamma/2$. We take as a fact that the Gaussian process (X) has a.s the modulus of continuity

$$\omega_X(h) = \mathcal{O}(\sqrt{|h|^{\gamma}\log(1/h)}) \tag{34}$$

on every compact subset of $\mathbb{R}^n$ (see [9, p.264]). Here the constant may depend on the realization of the process.
In some of the upcoming proofs, it is convenient to move the consideration into some abstract Hilbert space $\mathscr{H}$. Let us give a formal definition. It also explains why we choose to require completeness of the probability space.

Definition 2.35. *A metric space $\mathscr{H}$ with real- or complex inner product, complete with respect to the distance function induced by the inner product, is called a Hilbert space [13].*

Since we are only interested in its applications to probability, let us define the Gaussian Hilbert spaces. As a reminder, we assume that the expectation value of our Gaussian random variables (for recap, see (31)) is zero.

Definition 2.36. *A Gaussian Hilbert space is Gaussian linear space which is complete, i.e., a closed subspace of $L^2_{\mathbb{R}}(X, \Gamma, \mathcal{P})$ consisting of Gaussian random variables. [8]*

Example 2.37. *Take probability space $(\mathbb{R}, \overline{\mathcal{B}}, \mathcal{P})$, where $\overline{\mathcal{B}}$ is the completion of Borel σ-algebra and $d\mathcal{P} = (2\pi)^{-1}e^{-x^2/2}dx$. Then the function $\xi(x) = x$ is a Gaussian random variable and the set*

$$\mathscr{H} = \{tx : t \in \mathbb{R}\}$$

is a Gaussian Hilbert space.

If we have a Hilbert space H and a linear space G of Gaussian random variables, and mapping $f : H \to G$ is a linear isometry, that is a mapping between two metric spaces preserving distance, then $f(H)$ is called Gaussian Hilbert space indexed by H. If a Hilbert space $\mathscr{H}$ is given and G is a Gaussian linear space, then there is always a Gaussian Hilbert space indexed by $\mathscr{H}$ in G [8, Theorem 1.23]. More in-depth consideration of Hilbert spaces can be found in, for example, [8] for functional analysis or probability in mind.
In (n, d, γ)-process the coordinates of (X_t) are independent copies of $(n, 1, \gamma)$-process. Therefore, for our purposes, it is enough to show that $(n, 1, \gamma)$-process exists. This can be done as follows, which is adapted from [9]:
Take $\beta = \gamma/2$. We begin with a Gaussian β-helix indexed by $\mathbb{R}^n$, that is, a collection $\{X_j\}$ of functions of a Gaussian Hilbert space satisfying

$$||X_t - X_s|| = |t - s|^{\beta},$$

$0 < \beta < 1$, where $s, t \in \mathbb{R}^n$ and $(|| \cdot ||)$ is the norm of the Hilbert space. For construction of such, and some of its properties, see [9, Chapter 10, Section 5&6]. We write

$$|t|^{\gamma} = c \int_{\mathbb{R}^n} (1 - \cos(u \cdot t))|u|^{-n-\gamma} du,$$

where the constant $c = c(n, \gamma)$, $0 < \gamma < 2$, and consider function

$$Y_t(u) = e^{iu \cdot t} - 1, \; u, t \in \mathbb{R}^n. \tag{35}$$

Now $Y_t \in L^2(\mathbb{R}^n, c|u|^{-n-\gamma}du)$, and since $|e^{iu\cdot t} - e^{iu\cdot s}|^2 = 2[1-\cos(u\cdot(t-s))]$ we have

$$\int_{\mathbb{R}^n} |Y_t(u) - Y_s(u)|^2 \, c|u|^{-n-\gamma} du = 2|t-s|^\gamma.$$

Next, we turn the function Y_t into a complex valued $(n,1,\gamma)$-process by mapping $L^2(\mathbb{R}^n, c|u|^{-n-\gamma}du)$ into a complex Hilbert space $\mathscr{H}$ with a linear isometry. This can be done, for example, by choosing a linear mapping f which maps the orthonormal basis of $L^2(\mathbb{R}^n, c|u|^{-n-\gamma}du)$ to some basis of $\mathscr{H}$ [8, Example 1.22]. We get the real valued $(n,1,\gamma)$-process as follows: Let $Z_1(t), Z_2(t)$ be complex Gaussian $(n,1,\gamma)$-processes on $\mathscr{H}$. Set

$$X(t) = \frac{\sqrt{2}}{2}\left(\operatorname{Re} Z_1(t) + \operatorname{Im} Z_2(t)\right).$$

Then we only have to calculate

$$\begin{aligned}
&\mathbb{E}(|X(t) - X(s)|^2) \\
&= \frac{1}{2}\left[\mathbb{E}([\operatorname{Re} Z_1(t) - \operatorname{Re} Z_1(s)]^2) + \mathbb{E}([\operatorname{Im} Z_2(t) - \operatorname{Im} Z_2(s)]^2)\right] = |t-s|^\gamma.
\end{aligned}$$

The case $\gamma = 2$ follows by setting $X_t = Xt$. Now we get to the continuity of the process, the following theorem is called Dudley-Fernique theorem.

Theorem 2.38. *Suppose $K \subset \mathbb{R}^n$ is compact, let (X_t), $t \in K$, be a real valued Gaussian process indexed by K and $d(t,s) := ||X_t - X_s||$ a pseudodistance, that is a distance-like function, but not nescessarily point separating i.e $d(t,s) = 0 \not\Longrightarrow s = t$. If the integral*

$$J(K,d) = \int_0^\infty \sqrt{\log N(\varepsilon)} d\varepsilon, \quad (N(\varepsilon) = N_d(K,\varepsilon))$$

is finite, the process $(X_t), t \in K$, has an a.s continuous version $X(t,\omega)$, $(t \in K, \omega \in \Omega)$ and additionally

$$\left(\mathbb{E}(\sup_{t\in K} |X(t,\omega)|^p)\right)^{1/p} \leq C_p(J(K,d) + \inf_{t\in K} ||X_t||),\ p \geq 1 \tag{36}$$

where C_p is a function of p and $N(\varepsilon)$ is the smallest number of open balls of radius ε, with respect to d, that cover the set K.

Proof. See [9, Theorem 4, p.219]. □

A few words about the difference between Gaussian random variables and the (n, d, γ)-process. Gaussian random variable is a single random function defined on the probability space with expectation value and variance that happens to satisfy some conditions we gave to classify different random variables. On the other hand, (n, d, γ)-process is a collection $\{X_j\}_{j\in J}$ of elements of a Gaussian Hilbert space, a function $X : (J \subset \mathbb{R}^n) \to \{\text{measurable functions defined on}\, \mathbb{R}^d\}$. There are many ways to define the process, each with some own different properties. More information on the differences can be found in [8, Appendix B].

3 Fourier dimension

In this section, we get to the main topic of this study, the Fourier dimension. Before we write down the definition, let us give a little reasoning; By Theorem 2.31, the s-energy of a measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ can be calculated as

$$I_s(\mu) = \gamma(n, s) \int |\hat{\mu}(x)|^2 |x|^{s-n} dx.$$

A natural question to ask is, when does the measure μ have a finite s-energy? To initiate, one such condition would be that

$$|\hat{\mu}(x)| \leq |x|^{-s/2} \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{37}$$

Then $I_t(\mu)$ would be finite for all $t < s$ and, by Theorem 2.14, this would give that $\dim_{\mathrm{H}} \mathrm{spt}(\mu) \geq s$. Hence, if $\mu \in \mathcal{M}(A)$ and $\dim_{\mathrm{H}} A = s$, the best decay one may expect for the Fourier transform of μ at infinity is given by (37). There is also the question, whether one should limit the consideration only to Borel probability measures supported by A? If we only worked with compact sets this would not be a problem, but since in the last section of this study we aim to say something about the dimension of the image of a general Borel set of $\mathbb{R}$, we need to go the probability route. By the above consideration we give the following definitions:

Definition 3.1. *The Fourier dimension of set $A \subset \mathbb{R}^n$ is*

$$\dim_{\mathrm{F}} A = \sup \left\{ 0 \leq s \leq n : \exists\, \mu \in \mathcal{M}^1(A);\ |\hat{\mu}(\xi)| \leq C|\xi|^{-s/2} \text{ for all } \xi \in \mathbb{R}^n \right\},$$

for some constant $C > 0$.

Definition 3.2. *The Fourier dimension of a Borel probability measure μ on $\mathbb{R}^n$ is*

$$\dim_{\mathrm{F}} \mu = \sup \left\{ 0 \leq s \leq n : |\hat{\mu}(\xi)| \leq C|\xi|^{-s/2} \text{ for all } \xi \in \mathbb{R}^n \right\},$$

for some constant $C > 0$.

Definition 3.3. *Set $A \subset \mathbb{R}^n$ is a Salem set if for every $0 < s < \dim_{\mathrm{H}} A$ there is $\mu \in \mathcal{M}^1(A)$ such that $|\hat{\mu}(\xi)| \leq C|\xi|^{-s/2}$ for all $\xi \in \mathbb{R}^n$ for some constant $C > 0$.*

We shall dedicate the Chapter 4 to various Salem sets. From these definitions and by the consideration it follows that

$$\dim_{\mathrm{F}} A = \sup\left\{\dim_{\mathrm{F}} \mu : \mu \in \mathcal{M}^1(A)\right\}. \tag{38}$$

Before moving on, a short note about the restriction to Borel probability measures when we defined the Fourier dimension. For compact sets, one can normalize the Borel measures within its support, and then calculate the Fourier dimension with the given definition. But given general Borel set the definitions (with probability measures and without) wouldn't agree even if we considered finite unions of closed sets [10], so the restriction is necessary. There are also other variations to the definition of the Fourier dimension, for example, compact Fourier dimension and modified Fourier dimension, which takes into account some of the stability problems arising from the original definition. However, we do not consider the aforementioned variations in this study.

3.1 Properties of the Fourier dimension

Next, we introduce and prove some of the known properties of the Fourier dimension. The section is based on [2] and [3]. First off, the upper bound of Fourier dimension is given by Hausdorff dimension:

Proposition 3.4. *If $A \subset \mathbb{R}^n$ is a Borel set, then*

$$\dim_{\mathrm{F}} A \leq \dim_{\mathrm{H}} A.$$

Proof. Let $A \subset \mathbb{R}^n$ be a Borel set. If $\dim_{\mathrm{F}} A = 0$, the statement is trivial, so suppose that $0 < \dim_{\mathrm{F}} A$. Let $0 < t < \dim_{\mathrm{F}} A$. Now by Definition 3.1 there is a measure $\mu \in \mathcal{M}^1(A)$ such that

$$|\hat{\mu}(\xi)| \leq c|\xi|^{-t/2} \tag{39}$$

for all $\xi \in \mathbb{R}^n$ and for some positive and finite constant c. By Theorem 2.31 the t-energy of measure μ can be calculated as

$$I_t(\mu) = \gamma(n,t) \int |\hat{\mu}(x)|^2 |x|^{t-n} dx \leq C \int |x|^{-n} dt < \infty, \tag{40}$$

where the first inequality is due to (39). Hence by Theorem 2.16

$$\dim_{\mathrm{H}} A = \sup\{s : \exists\, \mu \in \mathcal{M}(A) \,\text{with}\, I_s(\mu) < \infty\} \geq \dim_{\mathrm{F}} A,$$

where the last inequality is due to (40). □

As a consequence of Proposition 3.4 and Definition 3.3, a Borel set $A \subset \mathbb{R}^n$ is a Salem set if and only if

$$\dim_{\mathrm{F}} A = \dim_{\mathrm{H}} A.$$

Next, we would like to see, whether Fourier dimension shares other properties with the Hausdorff dimension. Properties of the Hausdorff dimension include, for example, monotonicity and stability with respect to countable unions, ie. $\dim_{\mathrm{H}}(\bigcup_{n=1}^{\infty} A_i) = \sup_i \dim_{\mathrm{H}}(A_i)$ for $A_i \subset \mathbb{R}^n$, $i = 1, 2, \ldots$. Some hints about the latter has already been given.

Proposition 3.5. *The Fourier dimension of Borel sets is monotone.*

The following proof is adapted from [2].

Proof. Let $A \subset B \subset \mathbb{R}^n$ be Borel sets. Since $\mathcal{M}^1(A) \subset \mathcal{M}^1(B)$ we have

$$\begin{aligned} \dim_{\mathrm{F}} A = \sup\left\{\dim_{\mathrm{F}} \mu : \mu \in \mathcal{M}^1(A)\right\} &\leq \sup\left\{\dim_{\mathrm{F}} \mu : \mu \in \mathcal{M}^1(B)\right\} \\ &= \dim_{\mathrm{F}} B, \end{aligned}$$

as wanted. □

For Borel probability measures $\mu \ll \nu$, it follows from the Lebesgue's density theorem that $\underline{\dim}_{\mathrm{loc}}\mu(x) = \underline{\dim}_{\mathrm{loc}}\nu(x)$ for μ-a.e $x \in \mathbb{R}^n$. Thus the Hausdorff dimension of measure is monotone with respect to absolute continuity in a sense that

$$\mu \ll \nu \implies \dim_{\mathrm{H}} \nu \geq \dim_{\mathrm{H}} \mu \text{ and } \dim_{\mathrm{H}}^* \nu \geq \dim_{\mathrm{H}}^* \mu.$$

However, the same behaviour is not generally true for the Fourier dimension of measures. In proof of the following proposition we are going to construct a compact set $B \subset [0,1]$ with positive Lebesgue measure and zero Fourier dimension. Thus

$$\mathcal{L}^1_{|B} \ll \mathcal{L}^1_{|[0,1]} \ll \mathcal{L}^1_{|[0,1]} + \delta_0,$$

but

$$\dim_{\mathrm{F}} \mathcal{L}^1_{|B} = 0 \leq \dim_{\mathrm{F}} \mathcal{L}^1_{|[0,1]} = 1 \nleq \dim_{\mathrm{F}}(\mathcal{L}^1_{|[0,1]} + \delta_0) = 0. \tag{41}$$

Furthermore,

Proposition 3.6. *Fourier dimension is not countably stable.*

The following proof is adapted from [2, p.71 Lemma 6, p.72 Example 7] combined with notes from [3]. A shorter example of Fourier dimension not being countable stable was given in [3], but with this approach, we get the example (41).

Proof. We begin by proving the following statement: Let J be the set of positive integers and let $0 < \varepsilon \leq 1$. Then

$$\inf_{\mu} \sup_{j \in J} |\hat{\mu}(j)| \geq \frac{\varepsilon}{5}, \tag{42}$$

where the infimum is taken over measures $\mu \in \mathcal{M}^1([\varepsilon, 1])$;
Fix $\varepsilon > 0$ and let $\mu \in \mathcal{M}^1([\varepsilon, 1])$. If $\psi \in C_0([0, \varepsilon])$ is a function such that

$$\int \psi dx = 1 \text{ and } \sum_{k \in \mathbb{Z}} |\hat{\psi}(k)| < \infty,$$

we then have by Theorem 2.26

$$0 = \int \psi(x) d\mu(x) = \sum_{k \in \mathbb{Z}} \hat{\psi}(k) \overline{\hat{\mu}}(k) = 1 + 2 \operatorname{Re} \left(\sum_{k=1}^{\infty} \hat{\psi}(k) \overline{\hat{\mu}}(k) \right). \tag{43}$$

Rearranging the terms in (43), and followed by taking the norm we get

$$\frac{1}{2} \leq \sum_{k=1}^{\infty} |\hat{\psi}(k)| |\overline{\hat{\mu}}(k)| \leq \sup_{j \geq 1} |\overline{\hat{\mu}}(j)| \sum_{k=1}^{\infty} |\hat{\psi}(k)|. \tag{44}$$

Let $\chi(x) = \chi_{[0, \varepsilon/2]}(x)$ be the indicator function and choose the function ψ as the triangle function

$$\psi(x) = \left(\frac{2}{\varepsilon} \chi * \frac{2}{\varepsilon} \chi \right)(x).$$

Then by calculating the Fourier transform

$$\frac{2}{\varepsilon} \widehat{\chi}(\xi) = \frac{2}{\varepsilon} \int_0^{\varepsilon/2} e^{-i2\pi\xi x} dx = e^{-i\pi\varepsilon\xi/2} \frac{\sin(\pi\varepsilon\xi/2)}{\pi\varepsilon\xi/2} = e^{-i\pi\varepsilon\xi/2} \operatorname{sinc}(\varepsilon\xi/2) \tag{45}$$

we have by the convolution formula (5)

$$|\hat{\psi}(k)| = \left| \frac{2}{\varepsilon} \hat{\chi}(k) \right|^2 = \operatorname{sinc}^2 \left(\frac{k\varepsilon}{2} \right) \leq \min \left\{ 1, \frac{4}{k^2 \pi^2 \varepsilon^2} \right\}. \tag{46}$$

The last inequality in (46) is due to $\lim_{\xi \to 0} \operatorname{sinc}(x) = 1$ and for $x \neq 0$ $|\operatorname{sinc}(x)| \leq |x|^{-1}$. Next, we calculate upper bound for the sum $\sum_{k=1}^{\infty} |\hat{\psi}(k)|$. By (46)

$$\sum_{k=1}^{\infty} |\hat{\psi}(k)| \leq \left\lceil \frac{2}{\pi\varepsilon} \right\rceil + \frac{4}{\pi^2 \varepsilon^2} \sum_{\lceil \frac{2}{\pi\varepsilon} \rceil + 1}^{\infty} \frac{1}{k^2} \leq \frac{2 + \pi\varepsilon}{\pi\varepsilon} + \frac{4}{\pi^2 \varepsilon^2} \int_{\lceil \frac{2}{\pi\varepsilon} \rceil}^{\infty} x^{-2} dx \leq \frac{4 + \pi\varepsilon}{\pi\varepsilon}.$$

Therefore, applying the above estimate to (44) gives

$$\sup_{j\geq 1}|\overline{\hat{\mu}}(j)| \geq \frac{1}{2}\frac{\pi\varepsilon}{4+\pi\varepsilon} \geq \frac{\varepsilon}{5},$$

proving the first statement. Next, let $(l_k)_k$ be a strictly increasing sequence of natural numbers such that

$$\lim_{k\to\infty}\frac{l_k}{k} = \infty. \tag{47}$$

For each $k \in \mathbb{N}$ we define a compact set $A_k = \overline{\{x \in [0,1] : x_{l_k}\cdots x_{l_k+k} \neq 0^k\}}$, where $x = 0, x_1x_2\ldots$ is the binary expansion of x. Then for $n \in \mathbb{N}$ we can write

$$B_n = \bigcap_{k=n}^{\infty} A_k.$$

Now let $\mu \in \mathcal{M}^1(B_n)$, and for $k \in \mathbb{N}$ define a pushforward measure $\mu_k = f_{k*}\mu$, where $f_k(x) = 2^{l_k}x \mod 1$. Then if $k \geq n$, $\mu_k\big([2^{-k},1]\big) = 1$:
By the definition of a pushforward measure, $\mu_k\big([2^{-k},1]\big) = \mu\big(f_k^{-1}([2^{-k},1])\big)$. Now $x \in [2^{-k},1]$ if and only if the first k elements of its binary expansion are not identically zero, that is $x_1\cdots x_k \neq 0^k$. Therefore, if $x \in f_k^{-1}([2^{-k},1])$, then $2^{l_k}x \mod 1 \in [2^{-k},1]$ and $x_{l_k}\cdots x_{l_k+k} \neq 0^k$. Hence $B_n \subset f_k^{-1}([2^{-k},1])$, proving the claim since $k \geq n$.
By (42) there exists some $j_k \geq 1$ such that

$$\hat{\mu}(2^{l_k}j_k) = \hat{\mu}_k(j_k) \geq \frac{2^{-k}}{5}.$$

Therefore, for any $s > 0$,

$$\limsup_{\xi\to\infty}|\hat{\mu}(\xi)||\xi|^{s/2} \geq \lim_{k\to\infty}|\hat{\mu}(2^{l_k}j_k)||2^{l_k}j_k|^{s/2} \geq \lim_{k\to\infty}5^{-1}2^{\frac{1}{2}sl_k-k} = \infty,$$

where the last equality is due to (47), and hence we have that $\dim_{\mathrm{F}} B_n = 0$ for all $n \in \mathbb{N}$. Finally, to show that Fourier dimension is not countably stable, let $\nu = \mathcal{L}^1_{|[0,1]}$. Then for each n

$$\nu\left(\bigcup_{n=1}^{\infty} B_n\right) \geq \nu(B_n) \geq 1 - \sum_{k=n}^{\infty}2^{-k} = 1 - 2^{-(n+1)},$$

so that $\nu(B_n) \to 1$, as $n \to \infty$, and hence

$$\nu\left(\bigcup_{n=1}^{\infty} B_n\right) = 1,$$

giving us $\dim_{\mathrm{F}}(\bigcup_{n=1}^{\infty} B_n) = 1$, which we shall prove later on Chapter 4. $\square$

The example given in (41) works if one chooses $B = B_n$ like in the proof of Proposition 3.6 for any $n \in \mathbb{N}$ [3].

Proposition 3.7. *Let μ and ν be Borel probability measures on $\mathbb{R}^n$. Then*

$$\dim_{\mathrm{F}}(\mu + \nu) \geq \min\{\dim_{\mathrm{F}}\mu, \dim_{\mathrm{F}}\nu\}. \tag{48}$$

Proof. Let $0 < s < t \leq n$ and define a measure μ as $\mu = \mu_s + \mu_t$, where $|\hat{\mu}_s(x)| \leq C_1|x|^{-s/2}$ and $|\hat{\mu}_t(x)| \leq C_2|x|^{-t/2}$ for all $x \in \mathbb{R}^n$. Then, by the linearity of the Fourier transform and using the triangle inequality

$$\begin{aligned} |\hat{\mu}(x)| &= |\hat{\mu}_s(x) + \hat{\mu}_t(x)| \leq |\hat{\mu}_s(x)| + |\hat{\mu}_t(x)| \\ &\leq C_1|x|^{-s/2} + C_2|x|^{-t/2} = \mathcal{O}(|x|^{-s/2}), \end{aligned}$$

as $|x| \to \infty$. Thus $\dim_{\mathrm{F}}\mu \geq s$, proving the claim. □

Next, we are going to consider some cases where the inequality in (48) is equality. Before that, we need the following lemma.

Lemma 3.8. *Let μ be a finite Borel measure on $\mathbb{R}^n$ and let f be a non-negative C_0^m function, where $m = \lceil 3n/2 \rceil$ is given by the ceiling function. Define the measure ν on $\mathbb{R}^n$ by $d\nu = f d\mu$. If for all $0 \leq s \leq n$ and some $C > 0$*

$$|\hat{\mu}(\xi)| \leq C|\xi|^{-s/2} \quad \textit{for all } \xi \in \mathbb{R}^n, \tag{49}$$

then for all $0 \leq s \leq n$ and for some $\tilde{C} > 0$

$$|\hat{\nu}(\xi)| \leq \tilde{C}|\xi|^{-s/2} \quad \textit{for all } \xi \in \mathbb{R}^n.$$

In particular, $\dim_{\mathrm{F}}\nu \geq \dim_{\mathrm{F}}\mu$.

The following proof is adapted from [2, p.67 Lemma 1].

Proof. Since $f \in C_0^m$, there is a constant $M > 0$ such that for all $t \in \mathbb{R}^n$,

$$|\hat{f}(t)| \leq \frac{M}{1 + |t|^m}. \tag{50}$$

Because $\hat{f}$ is Lebesgue integrable, Theorem 2.21 holds pointwise everywhere for the function f, and we have by Fubini theorem

$$\begin{aligned} \hat{\nu}(t) &= \int e^{-i2\pi t\cdot\xi} f(\xi)d\mu(\xi) = \int e^{-i2\pi t\cdot\xi} \int e^{i2\pi\xi\cdot x}\hat{f}(x)dxd\mu(\xi) \\ &= \iint e^{-i2\pi\xi\cdot(t-x)}d\mu(\xi)\hat{f}(x)dx = \int \hat{\mu}(t-x)\hat{f}(x)dx. \end{aligned}$$

Estimating $|\hat{\nu}(t)|$ by dividing the integral into two parts, for $|t-x| < |t|/2$,

$$\begin{aligned}
&\int_{\{|t-x|<|t|/2\}} |\hat{\mu}(t-x)\hat{f}(x)|dx \leq \int_{\{|x|\geq|t|/2\}} \frac{\mu(\mathbb{R}^n)M}{1+|x|^m}dx \\
&\leq C' \int_{\{|x|\geq|t|/2\}} |x|^{-m}dx \leq C|t|^{n-m} \leq C|t|^{-n/2}, \qquad (51)
\end{aligned}$$

and for $|t-x| \geq |t|/2$, assuming (49) holds,

$$\int_{\{|x|\geq|t|/2\}} |\hat{\mu}(t-x)\hat{f}(x)|dx \leq \frac{C''2^{s/2}}{|t|^{s/2}} \int |\hat{f}(x)|dx \leq \tilde{C}|t|^{-s/2}. \qquad (52)$$

Combining (51) and (52), we get that whenever (49) holds,

$$|\hat{\nu}(t)| \leq C|t|^{-n/2} + \tilde{C}|t|^{-s/2} \leq \tilde{C}'|t|^{-s/2},$$

and thus $\dim_{\mathrm{F}} \nu \geq \dim_{\mathrm{F}} \mu$. □

Proposition 3.9. *Let μ and ν be finite Borel measures on $\mathbb{R}^n$ with disjoint compact supports. Then*

$$\dim_{\mathrm{F}}(\mu+\nu) = \min\{\dim_{\mathrm{F}} \mu, \dim_{\mathrm{F}} \nu\}.$$

The following proof is adapted from [2, p.77 Proposition 11].

Proof. Let $\phi \in C_0^\infty$ be a non-negative function such that $\phi(x) = 1$ for $x \in \operatorname{spt} \mu$ and $\phi(x) = 0$ for $x \in \operatorname{spt} \nu$. Then $\phi \cdot (\mu+\nu) = \phi\mu$ and by Lemma 3.8

$$\dim_{\mathrm{F}} \mu \geq \dim_{\mathrm{F}}(\mu+\nu).$$

On the other hand, let $\psi \in C_0^\infty$ be a non-negative function such that $\psi(x) = 1$ for $x \in \operatorname{spt} \nu$ and $\psi(x) = 0$ for $x \in \operatorname{spt} \mu$. Then $\psi \cdot (\mu+\nu) = \psi\mu$ and again, by Lemma 3.8

$$\dim_{\mathrm{F}} \nu \geq \dim_{\mathrm{F}}(\mu+\nu).$$

Thus,

$$\dim_{\mathrm{F}}(\mu+\nu) \leq \min\{\dim_{\mathrm{F}} \mu, \dim_{\mathrm{F}} \nu\},$$

and the claim follows from Proposition 3.7. □

Proposition 3.10. *Let μ be a finite Borel measure on $\mathbb{R}^n$ with compact support and let μ_a be a translation of μ by $a \in \mathbb{R}^n$. Then*

$$\dim_{\mathrm{F}}(\mu+\mu_a) = \dim_{\mathrm{F}} \mu.$$

The following proof is adapted from [2, p.77 Proposition 12].

Proof. Since
$$|\hat{\mu}_a(t)| = |e^{i2\pi a\cdot t}\hat{\mu}(t)| \leq |\hat{\mu}(t)|,$$
the Fourier dimension is translation invariant and thus,

$$\dim_{\mathrm{F}}(\mu + \mu_a) \geq \min\{\dim_{\mathrm{F}}\mu, \dim_{\mathrm{F}}\mu_a\} = \dim_{\mathrm{F}}\mu.$$

The opposite is clear if $t = 0$, so assume $t \neq 0$. Let k be a large enough odd integer such that $\operatorname{spt}\mu \cap \operatorname{spt}\mu_{ak} = \emptyset$. By linearity of the Fourier transform we have

$$|\widehat{\mu + \mu_a}(t)| \leq |1 + e^{i2\pi a\cdot t}||\hat{\mu}(t)| \leq 2|\cos(\pi a\cdot t)||\hat{\mu}(t)|.$$

The same calculation also gives

$$|\widehat{\mu + \mu_{ak}}(t)| \leq 2|\cos(\pi ak\cdot t)||\hat{\mu}(t)|.$$

Since k is odd, we can use L'Hôpital's rule to see that $\cos(kx)/\cos(x)$ is bounded by $|k|$, and therefore

$$|\widehat{\mu + \mu_{ak}}(t)| \leq 2\left|\frac{\cos(\pi ak\cdot t)}{\cos(\pi a\cdot t)}\right||\cos(\pi a\cdot t)||\hat{\mu}(t)| \leq C|\widehat{\mu + \mu_a}(t)|.$$

Hence by Proposition 3.9

$$\dim_{\mathrm{F}}(\mu + \mu_a) \leq \dim_{\mathrm{F}}(\mu + \mu_{ak}) = \min\{\dim_{\mathrm{F}}\mu, \dim_{\mathrm{F}}\mu_{ak}\} = \dim_{\mathrm{F}}\mu.$$

□

3.2 Sets of different Fourier and Hausdorff dimensions

Now that we know some of the properties of the Fourier dimension, we may begin to consider some examples. Like the title of this section could suggest, we look at certain sets and conditions under which the Fourier dimension and Hausdorff dimension do not agree in value. Hence we prove that they do not define the same property. By considering Cantor sets, we have the proof for wide range of self-similar sets. This section is based on [10, Chapter 8] with added notes from [12].

Cantor sets

We begin with a brief look at the construction. Let $0 < d < \frac{1}{2}$, and denote by I_0 the interval $[0,1] \subset \mathbb{R}$. From I_0, remove the middle open interval of length $1-2d$ and denote the remaining two new intervals as $I_{1,1}$ and $I_{1,2}$. We continue by removing the middle open intervals of length $(1-2d)d$ from $I_{1,1}$ and $I_{1,2}$, ending up with four intervals of length d^{-2}, denoted by $I_{2,i}, i = 1,2,3,4$. Inductively, at k:th step, remove the middle open intervals of length $(1-2d)d^{k-1}$, resulting in intervals $I_{k,i}, i = 1,\dots,2^k$ of length d^{-k}. The middle-d Cantor set is then defined as

$$C_d = \bigcap_{k=0}^{\infty} \bigcup_{i=1}^{2^k} I_{k_i}.$$

Define the natural measure $\mu_d \in \mathcal{M}(C_d)$ by setting

$$\mu_d(I_{k,i}) = 2^{-k} \quad \text{for all } k = 0,1,2,\dots,\ i = 1,\dots,2^k. \tag{53}$$

Choosing $s_d = \frac{\log 2}{\log(1/d)}$ it follows that

$$\mu_d = \mathcal{H}^{s_d}_{|C_d} \text{ and } \mathcal{H}^{s_d}(C_d) = 1. \tag{54}$$

The proof of the latter identity in (54) can be found on [10, p.110]. For the first one, by the definition of restriction of measure

$$\{A \subset \mathbb{R} :\ \mu_d(A) = 0\} = \left\{A \subset \mathbb{R} :\ \mathcal{H}^{s_d}_{|C_d}(A) = 0\right\},$$

so measures μ_d and $\mathcal{H}^{s_d}_{|C_d}$ are equivalent. To see that they are the same measure, we note that by (53) for all $k = 1,2,\dots, i = 1,\dots,2^k$ we have $\mu_d(I_{k,i}) = \operatorname{diam}(I_{k,i})^{s_d}$, so they also agree on value for all intervals (see [4, Theorem 1.14]). Then by Borel regularity this property can also be extended for all the Borel sets, thus giving us the claim.

Next, we are going to calculate the Fourier transform of μ_d. This is however much simpler if we use the following notations: First, C_d can be written in the form

$$C_d = \left\{ \sum_{j=1}^{\infty} \omega_j (1-d) d^{j-1} : \omega_j \in \{0,1\} \right\}.$$

Let $\Omega_k = \{(\omega_1,\dots,\omega_k) : \omega_i \in \{0,1\}\}$, and for all $\omega \in \Omega_k$, let

$$a(\omega) = \sum_{j=1}^{k} \omega_j (1-d) d^{j-1}.$$

Define a measure

$$\nu_k = 2^{-k} \sum_{\omega \in \Omega_k} \delta_{a(\omega)},$$

which converges weakly to the measure μ_d as $k \to \infty$. Now the Fourier transform of Dirac measure $\delta_{a(\omega)}$ is given by

$$\hat{\delta}_{a(\omega)}(u) = \int e^{-i2\pi ux} d\delta_{a(\omega)}(x) = e^{-i2\pi ua(\omega)},$$

so by linearity of the Fourier transform,

$$\hat{\nu}_k(u) = 2^{-k} \sum_{\omega \in \Omega_k} e^{-i2\pi ua(\omega)} = 2^{-k} \sum_{\omega \in \Omega_k} e^{i \sum_{j=1}^{k} \omega_j u_j}, \tag{55}$$

where $u_j = -2\pi(1-d)d^{j-1}u$. Opening the sums in the last expression of (55), we get that

$$\hat{\nu_k}(u) = 2^{-k} \prod_{j=1}^{k} (1 + e^{iu_j}).$$

Then calculating

$$(1 + e^{ix})/2 = e^{ix/2}(e^{ix/2} + e^{-ix/2})/2 = e^{ix/2} \cos(x/2),$$

we get that

$$\begin{aligned}
\hat{\nu}_k(u) &= \prod_{j=1}^{k} e^{iu_j/2} \cos(u_j/2) = \prod_{j=1}^{k} e^{iu_j/2} \prod_{j=1}^{k} \cos(u_j/2) \\
&= e^{i \sum_{j=1}^{k} u_j/2} \prod_{j=1}^{k} \cos(u_j/2).
\end{aligned}$$

In addition, from a further calculation

$$\begin{aligned}
\sum_{j=1}^{k} u_j/2 = \sum_{j=1}^{k} &-\pi(1-d)d^{j-1}u \\
&= -\pi \left[1 - d + (1-d)d + \ldots + (1-d)d^{k-1}\right] u \\
&= -\pi \left[1 - d + d - d^2 + d^2 - \ldots - d^{k-1} + d^{k-1} - d^k\right] u \\
&= -\pi(1 - d^k)u,
\end{aligned}$$

it follows that

$$\hat{\nu}_k(u) = e^{-i\pi(1-d^k)u} \prod_{j=1}^{k} \cos\left(\pi(1-d)d^{j-1}u\right),$$

which, as ν_k converges weakly to the measure μ_d, converges to

$$\hat{\mu}_d(u) = e^{-i\pi u} \prod_{j=1}^{\infty} \cos\left(\pi(1-d)d^{j-1}u\right), \tag{56}$$

as $k \to \infty$. Now pick, for example, $d = \frac{1}{3}$ which gives the classical middle third Cantor set. Then

$$\hat{\mu}_{\frac{1}{3}}(u) = e^{-i\pi u} \prod_{j=1}^{\infty} \cos\left(2\pi 3^{-j}u\right),$$

and choosing the sequence $u_k = 3^k$, $k \in \mathbb{N}$,

$$\hat{\mu}_{\frac{1}{3}}(3^k) = \prod_{j=1}^{\infty} \cos\left(2\pi 3^{k-j}\right) \not\to 0, \text{as } k \to \infty.$$

This gives that $\dim_{\mathrm{F}} \mu_{\frac{1}{3}} = 0 \neq \frac{\log 2}{\log 3} = \dim_{\mathrm{H}} \mu_{\frac{1}{3}}$. This doesn't yet imply that $\dim_{\mathrm{F}} C_{\frac{1}{3}} = 0$. The following theorem however does. First, let us prove a lemma.

Lemma 3.11. *Let $d^{-1} \geq 3$ be a integer. Denote by I the open interval $(d, 1-d)$ and $N = d^{-1}$. Then $[N^k x] \notin I$ for all $x \in C_d, k = 1, 2, \ldots$. Here $[y] \in [0,1)$, $y - [y] \in \mathbb{N}$, for all $y \geq 0$ denotes the fractional part.*

The following proof is adapted from [10, p.112].

Proof. Now all of the points $x \in C_d$ can be written in form

$$x = \sum_{j=1}^{\infty} \omega_j (1-d) d^{j-1} = \sum_{j=1}^{\infty} \omega_j (1 - N^{-1}) N^{1-j} = (N-1) \sum_{j=1}^{\infty} \omega_j N^{-j},$$

where $\omega_j \in \{1, 2\}$. Thus writing

$$\begin{aligned} N^k x &= (N-1) \sum_{j=1}^{\infty} \omega_j N^{k-j} \quad ||k - j := j \\ &= (N-1) \left[\sum_{j=0}^{k-1} \omega_{k-j} N^j + \sum_{j=1}^{\infty} \omega_{k+j} N^{-j} \right], \end{aligned}$$

and we have

$$[N^k x] = (N-1) \sum_{j=1}^{\infty} \omega_{k+j} N^{-j} \in C_d \subset [0,1] \setminus I.$$

$\square$

Theorem 3.12. *Let $d^{-1} \geq 3$ be an integer. Then for all measures $\mu \in \mathcal{M}(C_d)$*

$$\limsup_{|x| \to \infty} |\hat{\mu}(x)| > 0.$$

The following proof is adapted from [10, Theorem 8.1].

Proof. We shall give a proof by contradiction. Suppose $\mu \in \mathcal{M}(C_d)$ is such that $\hat{\mu}(k) \to 0$ as $|k| \to \infty$, $k \in \mathbb{Z}$. Choose a function $\phi \in \mathcal{S}(\mathbb{R})$ such that $\operatorname{spt} \phi \subset]d, 1-d[$ and $\int \phi dx = 1$. Again, let $N = d^{-1}$ and for $j = 1, 2, \ldots$ denote

$$\phi_j(x) = \phi([N^j x]), \ x \in [0,1].$$

From Lemma 3.11 we have $\operatorname{spt} \phi \cap C_d = \emptyset$, and since $\operatorname{spt} \phi \subset [0,1]$, by Theorem 2.24 $\phi(x) = \sum_{k \in \mathbb{Z}} \hat{\phi}(k) e^{i2\pi x k}, x \in [0,1]$, which gives us

$$\phi_j(x) = \sum_{k \in \mathbb{Z}} \hat{\phi}(k) e^{i2\pi x N^j k}, \ x \in [0,1],$$

and we see that $\hat{\phi}(N^j k) = \hat{\phi}(k)$. Furthermore, by Theorem 2.26

$$\begin{aligned} 0 = \int \phi_j d\mu &= \sum_{k \in \mathbb{Z}} \overline{\hat{\phi}_j(k)} \hat{\mu}(k) = \sum_{k \in \mathbb{Z}} \overline{\hat{\phi}_j(N^j k)} \hat{\mu}(N^j k) = \sum_{k \in \mathbb{Z}} \overline{\hat{\phi}_j(k)} \hat{\mu}(N^j k) \\ &= \hat{\phi}(0)\hat{\mu}(0) + \sum_{1 \leq |k| \leq m} \overline{\hat{\phi}_j(k)} \hat{\mu}(N^j k) + \sum_{|k| > m} \overline{\hat{\phi}_j(k)} \hat{\mu}(N^j k) \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Now $I_1 = \mu(C_d) > 0$. By assumption, for all $m \in \mathbb{N}$,

$$|I_2| = |\sum_{1 \leq |k| \leq m} \overline{\hat{\phi}_j(k)} \hat{\mu}(N^j k)| \leq 2m \sup_{|l| \geq N^j, l \in \mathbb{Z}} |\hat{\mu}(l)| \to 0,$$

as $j \to \infty$. And finally,

$$|I_3| = |\sum_{|k| > m} \overline{\hat{\phi}_j(k)} \hat{\mu}(N^j k)| \leq \mu(C_d) \sum_{|k| > m} |\hat{\phi}(k)|.$$

Since $\phi \in \mathcal{S}(\mathbb{R})$ we have $\widehat{\phi} \in \mathcal{S}(\mathbb{R})$, and hence there exists $m_0 \in \mathbb{N}$ such that

$$\sum_{|k| > m} |\hat{\phi}(k)| < \varepsilon,$$

for all $\varepsilon > 0$ and $m \geq m_0$. Thus $\mu(C_d) = 0$, which is a contradiction. $\square$

As a consequence, we have that $\dim_{\mathrm{F}}\mu = 0$ for all $\mu \in \mathcal{M}^1(C_{\frac{1}{d}})$, where $d \geq 3$ is an integer. Thus by definition, we have for example that

$$\dim_{\mathrm{F}} C_{\frac{1}{3}} = 0 \neq \frac{\log 2}{\log 3} = \dim_{\mathrm{H}} C_{\frac{1}{3}}.$$

This is the result that is commonly given to demonstrate that the Fourier dimension and Hausdorff dimension are not the same. More generally, the values of d for which $\hat{\mu}_d(u)$ doesn't tend to 0 at infinity can be characterized by introducing Pisot numbers. We say that a number $1 < \theta \in \mathbb{R}$ is a Pisot number if there exist $0 \neq \lambda \in \mathbb{R}$ such that

$$\sum_{k=0}^{\infty} \sin^2(\lambda\theta^k) < \infty. \tag{57}$$

Writing $\lambda\theta^k = \pi n_k + \delta_k$, where $n_k \in \mathbb{Z}$, $\frac{-\pi}{2} \leq \delta_k < \frac{\pi}{2}$ and using the standard formula $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, $\alpha, \beta \in \mathbb{R}$, we get that

$$\sin^2(\lambda\theta^k) = \sin^2(\pi n_k + \delta_k) = [(-1)^{n_k}\sin\delta_k]^2 = \sin^2\delta_k.$$

Since $0 \leq \sin^2 x \leq 1$ for all $x \in \mathbb{R}$, (57) comes to be equivalent with condition

$$\sum_{k=0}^{\infty} \delta_k^2 < \infty.$$

Theorem 3.13. *Let $0 < d < \frac{1}{2}$ and μ_d a Cantor measure. Then*

$$\lim_{u\to\infty} \hat{\mu}_d(u) = 0$$

if and only if $\frac{1}{d}$ is not a Pisot number.

The following proof is adapted from [10, Theorem 8.3].

Proof. Let $\theta = \frac{1}{d}$. First, suppose that

$$\hat{\mu}_d(u) \not\to 0 \text{ as } u \to \infty.$$

Then there exists $\delta > 0$ and an increasing sequence $(u_k)_k$ such that $u_k \to \infty$ and for all k

$$|\hat{\mu}_d(u_k)| > \delta.$$

Now write $\pi(1-d)u_k = \lambda_k\theta^{m_k}$, where $1 \leq \lambda_k < \theta$ and $(m_k)_k$ is an increasing sequence of positive integers. By changing the sequence $(\lambda_k)_k$ to a convergent subsequence if needed we may assume that $\lambda_k \to \lambda$, $1 \le \lambda < \theta$. Then by (56) we have

$$\delta < |\hat{\mu}_d(u)| = \left|\prod_{j=1}^{\infty} \cos\left(\pi(1-d)d^{j-1}u_k\right)\right| = \left|\prod_{j=1}^{\infty} \cos\left(\lambda_k \theta^{m_k-j+1}\right)\right| \\ \le \left|\prod_{j=0}^{m_k} \cos\left(\lambda_k \theta^{j}\right)\right|,$$

where the last inequality is due to $|\cos x| \le 1$ for all $x \in \mathbb{R}$. It follows that

$$\prod_{j=0}^{m_k} \left(1 - \sin^2\left(\lambda_k \theta^j\right)\right) \ge \delta^2.$$

Then by the inequality $x \le -\log(1-x)$ for $0 < x < 1$,

$$\sum_{j=1}^{m_k} \sin^2\left(\lambda_k \theta^j\right) \le \log(1/\delta^2).$$

For each integer $l > k$ we have

$$\sum_{j=1}^{m_k} \sin^2\left(\lambda_l \theta^j\right) \le \sum_{j=1}^{m_l} \sin^2\left(\lambda_l \theta^j\right) \le \log(1/\delta^2).$$

Fixing k and letting $l \to \infty$ leads to

$$\sum_{j=1}^{m_k} \sin^2\left(\lambda \theta^j\right) \le \log(1/\delta^2),$$

and letting $k \to \infty$

$$\sum_{j=1}^{\infty} \sin^2\left(\lambda \theta^j\right) \le \log(1/\delta^2).$$

Thus $\theta = \frac{1}{d}$ is a Pisot number. For the other direction, let $\theta = \frac{1}{d}$ be a Pisot number. Then there exists $\lambda \neq 0$ such that

$$\sum_{j=1}^{\infty} \sin^2\left(\lambda \theta^j\right) < \infty,$$

and therefore there exists $\varepsilon > 0$ for which

$$\sum_{j=1}^{\infty} \sin^2\left(\lambda \theta^j\right) \le \log(\varepsilon^{-2}).$$

Then like above we have

$$p := \prod_{j=0}^{\infty} |\cos\left(\lambda\theta^j\right)| \geq \varepsilon^{-2} > 0.$$

By choosing $u_k = \lambda\theta^k/(\pi(1-d))$,

$$\begin{aligned}|\hat{\mu}_d(u_k)| &= \left|\prod_{j=1}^{\infty} \cos\left(\lambda d^{j-1}\theta^k\right)\right| = \left|\prod_{j=1}^{k} \cos\left(\lambda\theta^j\right)\right| \left|\prod_{j=1}^{\infty} \cos\left(\lambda\theta^{-j}\right)\right| \\ &\geq p \left|\prod_{j=1}^{\infty} \cos\left(\lambda\theta^{-j}\right)\right| := pq,\end{aligned}$$

where $q > 0$, because from $\theta > 1$ it follows that $\sum_{j=1}^{\infty} \sin^2\left(\lambda\theta^{-j}\right) < \infty$ like above. Thus $\hat{\mu}_d$ doesn't tend to 0 at infinity proving the claim. $\square$

4 Salem sets

Salem sets are those with agreeing values of Fourier- and Hausdorff dimension. They are named after Greek mathematician Raphaël Salem who first gave an example of such in form of random construction in 1951. What is special about Salem sets is if you know the Fourier- or Hausdorff dimension of a given Salem set $A \subset \mathbb{R}^n$, say $\dim_{\mathrm{F}} A = t > 0$, then for $0 < s < t$ you can always find a measure $\mu \in \mathcal{M}^1(A)$ with finite s-energy satisfying

$$|\hat{\mu}(x)| \leq C|x|^{-s/2} \text{ for every } x \in \mathbb{R}^n,$$

for some constant C. We shall consider various examples of Salem sets from simple deterministic to ones with not so simple deterministic construction to random images. Trivial examples of Salem sets are sets of Hausdorff dimension zero. We have already encountered another one. By our choice of definition, $\mathcal{H}^1 = \mathcal{L}^1$, so in Proposition 3.6 we gave an example of a Salem set: the interval $[0,1] \subset \mathbb{R}$ (in the proof we wrote the interval as union of compact sets, $\bigcup_{n=1}^{\infty} B_n$). This can be seen by calculating

$$\left|\hat{\mathcal{L}}^1_{[0,1]}(\xi)\right| = \left|\int_{[0,1]} e^{-i2\pi xu} du\right| \leq \frac{1}{\pi|x|}|e^{-i\pi x}\sin(\pi x)| \leq \frac{1}{\pi|x|}\sqrt{\pi|x|} = \mathcal{O}(|x|^{-1/2})$$

giving us that $\dim_{\mathrm{F}}([0,1]) \geq 1$ on $\mathbb{R}$, hence the result. This also gives Example 2.29 in the case $n = 1$.

4.1 Deterministic Salem sets

$S^{\text{n-1}}$

We consider one more concrete Salem set, the unit sphere in $\mathbb{R}^n$, where $n \geq 2$. Let us define a measure

$$\mu_\delta = \delta^{-1}\, \mathcal{L}^{\text{n}}_{|\{B(0,1+\delta)\setminus B(0,1)\}},$$

which converges weakly into surface measure σ^{n-1} as $\delta \to 0$. Then by changing the variables, $s := |t|s$,

$$\begin{aligned}\hat{\mu}_\delta(t) &= c(n)|t|^{-(n-2)/2} \int_0^\infty \delta^{-1}\chi_{\{B(0,1+\delta)\setminus B(0,1)\}}(s) J_{(n-2)/2}(2\pi|t|s)s^{n/2} d\mathcal{L}^{\text{n}}(s) \\ &= c|t|^{-n} \int_{|t|}^{(1+\delta)|t|} \delta^{-1} J_{(n-2)/2}(2\pi s)s^{n/2} d\mathcal{L}^{\text{n}}(s). \qquad (58)\end{aligned}$$

We estimate (58) by using property 1) of Bessel functions giving us

$$\begin{aligned}|\hat{\mu}_\delta(t)| &\leq C|t|^{-n} \int_{|t|}^{(1+\delta)|t|} \delta^{-1} s^{(n-1)/2} d\mathcal{L}^{\text{n}}(s) \\ &= C(n)|t|^{-(n-1)/2} \left[((1+\delta)^{(n+1)/2} - 1)/\delta\right].\end{aligned}$$

Then, by L'Hôpital's rule,

$$\begin{aligned}\lim_{\delta\to 0} |\hat{\mu}_\delta(t)| &= C|t|^{-(n-1)/2} \lim_{\delta\to 0} \left[((1+\delta)^{(n+1)/2} - 1)/\delta\right] \\ &= C(n)|t|^{-(n-1)/2} \lim_{\delta\to 0}(1+\delta)^{(n+1)/2-1} = C(n)|t|^{-(n-1)/2}.\end{aligned}$$

Thus $|\hat{\sigma}^{n-1}(t)| \leq C(n)|t|^{-(n-1)/2}$ for all $t \in \mathbb{R}^n$. Hence $\dim_{\mathrm{F}} S^{\text{n-1}} \geq n-1$ on $\mathbb{R}^n$ and because $\dim_{\mathrm{H}} S^{\text{n-1}} = n-1$, on $\mathbb{R}^n$

$$\dim_{\mathrm{F}} S^{\text{n-1}} = \dim_{\mathrm{H}} S^{\text{n-1}}.$$

Hence $S^{\text{n-1}}$ is a Salem set on $\mathbb{R}^n$. However, Fourier dimension depends on the ambiet space. For example, take $S^1 \times \{0\} \subset \mathbb{R}^3$, but for $\nu \in \mathcal{M}(S^1 \times \{0\})$ and $(s,t) \in S^1 \times \mathbb{R}$

$$\widehat{\nu}(s,t) = \widehat{\nu}(s,0),$$

and therefore $\widehat{\nu}(s,t) \not\to \overline{0}$ as $t \to \infty$, if $\widehat{\nu}(s,0) \neq \overline{0}$.

4.1.1 Cantor-type sets

This section is based on Bluhm's paper on a theorem of Kaufman; that is, the set of α-well approximable numbers is a deterministic Salem set of dimension $2/(2+\alpha)$. Here $\alpha > 0$ can be chosen to produce a Salem set of any dimension strictly between 0 and 1. Bluhm's work is a modification to Kaufman's approach, however, deterministic Salem sets of any given dimension strictly between 0 and 1 can be constructed by it. We shall divide the proof into smaller parts.

In this section, let $||x||$ denote the distance of $x \in \mathbb{R}$ to the nearest integer,

$$||x|| = \min_{k\in\mathbb{Z}} |x-k|.$$

For a positive integer M, let $\mathbb{P}_M = \mathbb{P} \cap [M, 2M]$, where $\mathbb{P}$ denotes the set of prime numbers. The set of numbers we shall be working with is

$$E_\alpha = \bigcap_{k=1}^{\infty} \bigcup_{p\in\mathbb{P}_{M_k}} \overline{E}_p(\alpha), \tag{59}$$

where for a fixed $\alpha > 0$, $(M_k)_k$ is a sequence constructed later such that

$$M_1 < 2M_1 < M_2 < 2M_2 < M_3 < \dots, \tag{60}$$

and for every $q \in \mathbb{N}$

$$\begin{aligned} \overline{E}_q(\alpha) &= \left\{x \in [0,1] : ||qx|| \le q^{-1-\alpha}\right\} \\ &= \left[0, q^{-2-\alpha}\right] \cup \bigcup_{m=1}^{q-1} \left[\frac{m}{q} - q^{-2-\alpha}, \frac{m}{q} + q^{-2-\alpha}\right] \cup \left[1 - q^{-2-\alpha}, 1\right]. \end{aligned}$$

Thus, as an intersection of countable unions of closed sets, E_α is a compact set. Due to equation [7, 22.19.3], which tells that, for $x \ge 1$, the number of prime numbers on the interval $[x, 2x]$ is approximately the same as on the interval $[0, x]$, we have the prime number theorem

$$\lim_{M\to\infty} \frac{\#\mathbb{P}_M}{M/\log M} = 1, \tag{61}$$

and thus we may find a sequence $(M_k)_k$ satisfying the following condition; Let $M_1 \in \mathbb{N}$ be large enough such that for every $k \in \mathbb{N}$,

$$\mathbb{P}_{M_k} \neq \emptyset \quad \text{and } \#\mathbb{P}_{M_k} \ge \frac{M_k}{2\log M_k}. \tag{62}$$

From now on, let us assume that the condition (62) is fulfilled. It follows that $\{0,1\} \in \overline{E}_p(\alpha)$ for all $p \in \mathbb{P}_{M_k}$, $k \in \mathbb{N}$ and hence E_α is a non-empty compact set. Next, we consider the following proposition:

Proposition 4.1. *E_α supports a finite h-measure with a gauge function*

$$h(x) = x^{2/(2+\alpha)} \log(e + x^{-1}).$$

The following proof is adapted from [1, Proposition 2.2].

Proof. Let $q \in \mathbb{N}$. The set $\overline{E}_q(\alpha)$ can be covered with $q-1$ intervals of length $a = 2q^{-2-\alpha}$. Thus

$$\begin{aligned}\mathcal{H}^{\mathrm{h}}_a(\overline{E}_q(\alpha)) &\leq (q-1)h(a) = (q-1)(2q^{-2-\alpha})^{2/(2+\alpha)} \log(e + (2q^{-2-\alpha})^{-1}) \\ &\leq c(\alpha) q^{-2(1+\alpha)/(2+\alpha)} \log(e + \frac{1}{2} q^{2+\alpha}). \end{aligned} \tag{63}$$

Now $E_\alpha \subset \bigcup_{p \in \mathbb{P}_{M_k}} \overline{E}_p(\alpha)$ for all $k \in \mathbb{N}$, so

$$\begin{aligned}\mathcal{H}^{\mathrm{h}}_a(E_\alpha) &\leq \limsup_{k\to\infty} \mathcal{H}^{\mathrm{h}}_a\left(\bigcup_{p\in\mathbb{P}_{M_k}} \overline{E}_p(\alpha)\right) \leq \limsup_{k\to\infty} \sum_{p\in\mathbb{P}_{M_k}} \mathcal{H}^{\mathrm{h}}_a\left(\overline{E}_p(\alpha)\right) \\ &\leq \limsup_{k\to\infty} \#\mathbb{P}_{M_k} \max_{p\in\mathbb{P}_{M_k}} \left[c(\alpha) p^{-2(1+\alpha)/(2+\alpha)} \log(e + \frac{1}{2}p^{2+\alpha})\right], \end{aligned} \tag{64}$$

where the last inequality is due to (63). Now for $p \in \mathbb{P}_{M_k}$

$$\log(e + \frac{1}{2}p^{2+\alpha}) \leq \log(e + \frac{1}{2}(2M_k)^{2+\alpha}) \leq c(\alpha)\log(M_k). \tag{65}$$

On the other hand for $p \in \mathbb{P}_{M_k}$ we have

$$p^{2(1+\alpha)/(2+\alpha)} \geq (M_k)^{(1+\alpha)/(1+\frac{\alpha}{2})} \geq M_k. \tag{66}$$

Therefore applying (65) and (66) to (64) we get

$$\mathcal{H}^{\mathrm{h}}_a(E_\alpha) \leq \limsup_{k\to\infty} C(\alpha) \frac{\#\mathbb{P}_{M_k}}{M_k/\log(M_k)} = C(\alpha),$$

where last equality due to (61), hence proving the proposition. $\square$

As a consequence of Proposition 3.4, $\dim_{\mathrm{F}} E_\alpha \leq \dim_{\mathrm{H}} E_\alpha \leq 2/(2+\alpha)$. Next we would like to construct a measure μ_α on the set E_α for which $\dim_{\mathrm{F}} \mu_\alpha \geq 2/(2+\alpha)$. Before that we need to introduce some notation.

Fix $M \in \mathbb{N}$, for which we write $R = (4M)^{-1-\alpha}$. On the interval $[-\frac{1}{2}, \frac{1}{2}]$, define a function F_M,

$$F_M(x) = \begin{cases} \frac{15}{16} R^{-5}(R^2 - x^2)^2, & |x| \leq R \\ 0, & R < |x| \leq \frac{1}{2}. \end{cases}$$

From now on, we assume that F_M is defined on the whole line $\mathbb{R}$ as 1-periodic function. Because $F_M \in C^2$, its Fourier series

$$F_M(x) = \sum_{k\in\mathbb{Z}} a_k^{(M)} e^{i2\pi kx}$$

converges uniformly, when the coefficients are given by

$$a_k^{(M)} = \int_{-\frac{1}{2}}^{\frac{1}{2}} F_M(t)e^{-i2\pi kt}dt.$$

Simple calculation gives that $a_0^{(M)} = 1$ and therefore $|a_k^{(M)}| \leq 1$ for all k. Furthermore, integration by parts three times also gives that $|a_k^{(M)}| \leq k^{-2}R^{-2}$ for all integers $k \geq 1$. Next, define a function

$$q_M(x) = \sum_{p\in\mathbb{P}_M} F_M(px) = \sum_{k\in\mathbb{Z}} \sum_{p\in\mathbb{P}_M} a_k^{(M)} e^{i2\pi kpx}.$$

Now, we have the Fourier transform

$$\hat{q}_M(m) = \sum_{k\in\mathbb{Z}, p\in\mathbb{P}_m, m=pk} a_k^{(M)}, \tag{67}$$

so $c_M = (\#\mathbb{P}_M)^{-1}$ is a normalizing constant such that $c_M\hat{q}_M(0) = 1$. Denote by $g_M := c_M q_M$. Again, $g_M \in C^2$ and it is a 1-periodic function. We note that if $g_M(x) > 0$, there exists $p \in \mathbb{P}_m$ for which

$$||px|| \leq p^{-1-\alpha} :$$

This is due to F_M being 1-periodic. If $g_m(x) > 0$, then there are $p \in \mathbb{P}_M$ and $k \in \mathbb{Z}$ such that $|px - k| \leq R = (4M)^{-1-\alpha}$. Finally, let

$$\theta(x) = (1 + |x|)^{-1/(2+\alpha)} \log(e + |x|) \log\log(e + |x|).$$

Lemma 4.2. *For every $\psi \in C_0^2$ and $\delta > 0$ there exists a positive integer $M_0 = M_0(\psi, \delta)$ such that for $x \in \mathbb{R}$ and for all $M \geq M_0$,*

$$|\widehat{\psi g}_M(x) - \hat{\psi}(x)| \leq \delta\theta(x).$$

The following proof is adapted from [1, Lemma 3.2].

Proof. Let $\delta > 0$. For now, fix $M \in \mathbb{N}$. By (67) and $|a_k^{(M)}| \leq 1$ it follows that

$$|\hat{q}_M(m)| \leq \#\{(k, p) \in \mathbb{Z} \times \mathbb{P}_M : m = kp\}.$$

Clearly, if $|m| < M$, $|\hat{q}_M(m)| = 0$ since there is no integer k for which $m = kp$ when $p \in \mathbb{P}_M$. On the other hand, for every $m \in \mathbb{Z} \setminus \{0\}$, $|m|$ has unique prime factorization

$$|m| = p_1^{a_1} p_2^{a_n} \cdots p_n^{a_n} \geq M^{\#\{\text{prime factors of } |m| \text{ in } [M,2M]\}},$$

so we have

$$|\hat{q}_M(m)| \leq \frac{\log |m|}{\log M}. \tag{68}$$

In addition, since $m = kp$, for $p \in \mathbb{P}_M$ we have $|m| = \frac{|k|}{p} \geq \frac{|k|}{2M}$, and $|a_k^{(M)}| \leq k^{-2} R^{-2} \leq 4m^2 M^2 R^{-2}$ for all $m \in \mathbb{Z} \setminus \{0\}$, and hence

$$|\hat{q}_M(m)| \leq \frac{4m^2 M^2 R^{-2} \log |m|}{\log M}. \tag{69}$$

We shall consider the rest of the proof in three parts. The first part of the proof: There exists $M_1 > 0$ and $A = A(\alpha) > 0$ such that for all $M \geq M_1$,

$$\begin{cases} |\hat{g}_M(m)| \leq A M^{-1} \log M, & \text{for all } m \in \mathbb{Z} \setminus \{0\} \\ |\hat{g}_M(m)| \leq A |m|^{-1/(2+\alpha)} \log |m|, & \text{for all } m \in \mathbb{Z} \text{ with } |m| > 4MR^{-1}. \end{cases} \tag{70}$$

Let $1 \leq |m| \leq 4MR^{-1}$. By (68) and the definition of R,

$$\begin{aligned} |\hat{g}_M(m)| &= c_M |\hat{q}_M(m)| \leq \frac{c_M \log |m|}{\log M} \leq \frac{2M^{-1} \log M (\log(4M) - \log R)}{\log M} \\ &= 2M^{-1}(\log 4 + \log(M) + (1+\alpha)(\log 4 + \log(M))) \leq 4(2+\alpha) M^{-1} \log M. \end{aligned}$$

For $|m| > 4MR^{-1} = (4M)^{2+\alpha}$, by (69)

$$\begin{aligned} |\hat{g}_M(m)| &\leq \frac{c_M 4m^{-2} M^2 R^{-2} \log |m|}{\log M} \leq \frac{2M^{-1} \log(M) 4m^{-2} M^2 R^{-2} \log |m|}{\log M} \\ &= 8m^{-2} M R^{-2} \log |m| = 8m^{-2} \frac{1}{4} (4M)^{3+2\alpha} \log |m| \leq 2|m|^{-1/(2+\alpha)} \log |m|. \end{aligned}$$

To finish the first inequality in (70), for $|k| > 4MR^{-1}$ and $M \geq M_1$

$$\begin{aligned} |\hat{g}_M(m)| &\leq 2|m|^{-1/(2+\alpha)} \log |m| \leq 2(4MR^{-1})^{-1/(2+\alpha)} (\log(4M) - \log R) \\ &= 2(4M(4M)^{-1-\alpha})^{-1/(2+\alpha)} (\log 4 + \log M + (1+\alpha)(\log 4 + \log M)) \\ &\leq 2(4M)^{-\alpha/(2+\alpha)} 4(\alpha+2) \log M \leq 2(2+\alpha) M^{-1} \log M, \end{aligned}$$

which proves the first part with $A = 2(\alpha+2)$. From now on, $M \geq M_1$ and let $\psi \in C_0^2$ be given.

The second part of the proof: There exists a constant $B = B(\psi, \alpha) > 0$ such that

$$|\widehat{\psi g}_M(x) - \hat{\psi}(x)| \leq BM^{-1} \log M \quad \text{for } x \in \mathbb{R}. \tag{71}$$

The Fourier series of function $\widehat{\psi g}_M$ can be written as

$$\widehat{\psi g}_M(x) = \sum_{m \in \mathbb{Z}} \hat{g}_M(m) \hat{\psi}(x - m).$$

Because $\psi \in C_0^2$, we have $|\hat{\psi}(\xi)| \leq B_1(\psi)(1 + |\xi|)^{-2}$ for all $\xi \in \mathbb{R}$, and since $\hat{g}_M(0) = 1$, applying (70) we get

$$\begin{aligned}
|\widehat{\psi g}_M(x) - \hat{\psi}(x)| &\leq \sum_{m \neq 0} |\hat{g}_M(m)||\hat{\psi}(x - m)| \leq B_1 \sum_{m \neq 0} |\hat{g}_M(m)|(1 + |x - m|)^{-2} \\
&\leq B_1 \sum_{m \neq 0} (1 + |x - m|)^{-2} \sup_{m \neq 0} |\hat{g}_M(m)| \leq 2AB_1 M^{-1} \log M \sum_{m=1}^{\infty} m^{-2} \\
&= BM^{-1} \log M. \qquad (72)
\end{aligned}$$

Here the constant $B = \frac{\pi^2}{3} AB_1$, where A is the constant calculated in the first part.

The third part of the proof: There exists $M_2 > 0$ such that for all $M \geq M_2$

$$|\widehat{\psi g}_M(x) - \hat{\psi}(x)| \leq \delta\theta(x) \quad \text{for } x \in \mathbb{R}. \tag{73}$$

First, we consider the case when $|x| < 8MR^{-1}$. By (71), the left-hand side of the inequality (73) is bounded by a constant depending only on ψ, α, and M. Since $M^{-1} \log(M)$ tends to zero as the value of M increases, and $\theta(x) \geq 0$, we may choose $M' = M'(\psi, \delta)$ big enough such that inequality (73) holds.

In the case $|x| \geq 8MR^{-1}$, we can obtain a better estimate. Again, by the calculation leading to (72), for fixed $x \in \mathbb{R}$ we write

$$|\widehat{\psi g}_M(x) - \hat{\psi}(x)| \leq B_1 \sum_{m \neq 0} |\hat{g}_M(m)|(1 + |x - m|)^{-2} := I_1 + I_2,$$

where the consideration is divided so that I_1 is the sum over those m for which $|x - m| \leq |x|/2$ and I_2 is the sum over those m for which $|x - m| > |x|/2$. Then

$$\begin{aligned}
I_1 &\leq \sum_{|x-m| \geq \frac{|x|}{2}} B_1 |\hat{g}_M(m)|(1 + |x - m|)^{-2} \\
&\leq B_1 M^{-1} \log(M) \sum_{|x-m| \geq \frac{|x|}{2}} (1 + |x - m|)^{-2} \leq C|x|^{-1},
\end{aligned}$$

where C is a constant independent of x and M, and

$$I_2 = \sum_{|x-m|<\frac{|x|}{2}} B_1|\hat{g}_M(m)|(1+|x-m|)^{-2} \le \left(2B_1 \sum_{m=1}^{\infty} m^{-2}\right) \sup_{\frac{|x|}{2}<|m|} |\hat{g}_M(m)|$$
$$\le B \sup_{\frac{|x|}{2}<|m|} |m|^{-1/(2+\alpha)} \log|m| \le \delta\theta(x), \tag{74}$$

for $M \ge M''$. Constant $M'' = M''(\psi, \delta)$ satisfying (74) can be found since $\sup_{\frac{|x|}{2}<|m|} |m|^{-1/(2+\alpha)} \log|m|$ is decreasing as the value of M increases. The inequality in (73) then holds by choosing $M_2 = \max\{M', M''\}$.
Hence the lemma is proved by choosing constant $M_0(\psi, \delta) = M_2$. □

Next, we can construct the sequence $(M_k)_k$ we mentioned in (60):
Let $\psi_0 \in C_0^2$ be a function such that

$$\int \psi_0(x)dx = 1, \quad \psi_{0|]0,1[} > 0, \quad \psi_{0|\mathbb{R}\setminus]0,1[} \equiv 0.$$

Next, choose $0 < \tau < \frac{1}{2}$ and write $\delta_k = \tau 2^{-k}, k \in \mathbb{N}$. By using Lemma 4.2 inductively we find

$$\begin{cases} M_1 = M_1(\psi_0, \tau 2^{-1}) \\ M_2 = M_2(\psi_0 g_{M_1}, \tau 2^{-2}) \\ M_3 = M_3(\psi_0 g_{M_1} g_{M_2}, \tau 2^{-3}) \\ \vdots \\ M_k = M_k(\psi_0 g_{M_1} g_{M_2} \cdots g_{M_{k-1}}, \tau 2^{-k}), k \in \mathbb{N}. \end{cases}$$

From now on, we assume that the set E_α (see (59)) is constructed according to sequence $(M_k)_k$ given above. Next, we define functions

$$G_0 := 1, \quad G_k = \prod_{m=1}^{k} g_{M_m}, k \in \mathbb{N}.$$

Again, by using Lemma 4.2 inductively, we obtain for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$

$$|\widehat{\psi_0 G}_{k+1}(x) - \widehat{\psi_0 G}_k(x)| \le \tau 2^{-k-1}\theta(x). \tag{75}$$

Next, we define a sequence of measures $(\mu_k)_k$ by setting for all $k \in \mathbb{N}$

$$\mu_k = \psi_0 G_k \mathcal{L}^1,$$

with the Fourier transforms

$$\widehat{\mu}_k(t) = \widehat{\psi_0 G_k}(t).$$

By inequality (75) we have $(\mu_k)_k$ is a Cauchy sequence with respect to the supremum norm and is therefore bounded. Hence by Theorem 2.11 there exists a measure $\mu_\alpha \in \mathcal{M}([0,1])$ such that $c(\tau)\mu_k \to \mu_\alpha$ weakly as $k \to \infty$, where $c(\tau)$ is a normalization constant. The following theorem finishes this section:

Theorem 4.3. *Measure μ_α satisfies*

$$\widehat{\mu}_\alpha(x) = \mathcal{O}(\theta(x)).$$

Therefore, $\dim_{\mathrm{F}} E_\alpha \geq 2/(2+\alpha)$.

The following proof is adapted from [1, Theorem 3.3].

Proof. First, for each $k \in \mathbb{N}$, the closed support of μ_α is contained in the closure of set

$$E_k := \{x \in \mathbb{R} : \psi_0(x) G_k(x) > 0\}.$$

Since $G_k(x) > 0$ for $x \in E_\alpha$ and $\psi_0(x) > 0$ for $x \in]0,1[$, and because E_α is compact,

$$\bigcap_{k=1}^{\infty} \overline{E}_k \subset E_\alpha.$$

Therefore

$$\mu_\alpha \in \mathcal{M}^1(E_\alpha).$$

By (75),

$$|\hat{\mu}_k(x)| \leq C\theta(x)$$

for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$, where $C > 0$ is a constant, so

$$|\hat{\mu}_\alpha(x)| \leq \tilde{C}\theta(x)$$

for all $x \in \mathbb{R}$ and some constant $\tilde{C} > 0$, proving the asymptotic. Further considering the asymptotic behaviour of $\theta(x)$ by calculating the limit as $|x| \to \infty$ using L'Hôpitals rule, we obtain that $\theta(x) = \mathcal{O}(|x|^{-1/(2+\alpha)})$. Thus $\dim_{\mathrm{F}} E_\alpha \geq 2/(2+\alpha)$. $\square$

Combining Theorem 4.3 with the consequence of Proposition 4.1, we have

$$\dim_{\mathrm{F}} E_\alpha = \dim_{\mathrm{H}} E_\alpha = 2/(2+\alpha).$$

Thus we have an example of a non-trivial deterministic Salem set.

4.2 Random Salem sets

4.2.1 Images of linear sets and measures under Brownian motion

In this section, we are going to prove that a compact set of Hausdorff dimension $\alpha < \frac{n}{2}$ on a line under a sample function of n-dimensional Brownian motion is almost surely a Salem set of dimension 2α. We shall divide the proof into smaller parts, starting by defining some random series and ending with the result. This section is based on Kahane's book [9].

Fourier-Wiener series

Wiener process, or Brownian motion, is an important example of a Gaussian process. The construction doesn't differ much from the one we introduced in the preliminaries and can be found, for example, in [9, p.233]. Instead of deriving the same result again we introduce the Wiener function through Fourier-Wiener series, which when convergent, is the a.s continuous version of the process, the existence of which follows from the Dudley-Fernique theorem. Let $X_0, X_1, \dots, Y_1, Y_2, \dots$ be a subnormal sequence on $\mathbb{R}^n$ such that for all $k = 1, 2, \dots$

$$\mathbb{E}(|X_0|^2) = \mathbb{E}(|X_k|^2) = \mathbb{E}(|Y_k|^2) = n.$$

For $t \geq 0$, define the n-dimensional Wiener function W as

$$W(t) = X_0 t + \sum_{k=1}^{\infty} \frac{\sqrt{2}}{2\pi k} \left[X_k \sin(2\pi k t) + Y_k (1 - \cos(2\pi k t)) \right]. \tag{76}$$

Now $W(t) = (W_1(t), W_2(t), \dots, W_n(t))$, where W_k are independent for all $k = 1, \dots, n$. At first glance, this function does not look that well-behaving. We are going to first prove that the function W represents a continuous function almost surely. For clearer presentation, let us use the following notation:

$$s_j = \left(2 \sum_{2^j \leq k < 2^{j+1}} a_k^2 \right)^{\frac{1}{2}}, j = 0, 1, 2, \dots, \tag{77}$$

where $a_k = \frac{\sqrt{2}}{2\pi k}$.

Definition 4.4. *We say that function P is a random trigonometric polynomial of degree $N \in \mathbb{N}$ if it's of the form*

$$P(t) = \sum_{k=1}^{N} X_k a_k \cos(t + \phi_k),$$

where $(X_k)_k$ is a subnormal sequence and a_k, ϕ_k are some given real numbers.

The following lemma is used to estimate random trigonometric polynomials in proofs of some of the following theorems.

Lemma 4.5. *Let $P(t) = \sum \xi_k f_k(t)$ be a random trigonometric polynomial, where f_k are real valued trigonometric polynomials of degree less than or equal to $N \in \mathbb{N}$ defined on the circle, $(\xi_k)_k$ is a subnormal sequence, and the sum is finite. Then*

$$\mathcal{P}\left(||P||_\infty \geq C \left(\sum_k ||f_k||_\infty^2 \log N \right)^{\frac{1}{2}} \right) \leq \frac{1}{N^2}, \tag{78}$$

for some constant $C > 0$.

The proof is adapted from [9, p.68 Theorem 1, p.69 Theorem 2, p.49 Proposition 5].

Proof. Let us denote the circle by E and let μ be a measure with $\mu(E) < \infty$. Let B be the set of all trigonometric polynomials of degree less or equal to N defined on E. Also, suppose that there exists $\rho \geq 1$ with the following property: If $f \in B$, then there is an interval $I = I(f) \subset E$ with $\mu(I) \geq \frac{\mu(E)}{\rho}$ and

$$|f(t)| \geq \frac{1}{2} ||f||_\infty \text{ for } t \in I. \tag{79}$$

First, let $f_k \in B$ and $(\xi_k)_k, k \in \mathbb{N}$ be a subnormal sequence. Denote by

$$r = \sum ||f_k||_\infty^2, \;\; M = ||P||_\infty.$$

For a fixed $-\infty < \lambda < \infty$ we then have the expectation value

$$\mathbb{E}\left(e^{\lambda P(t)}\right) = \mathbb{E}\left(\prod_k e^{\lambda \xi_k f_k(t)} \right) = \prod_k \mathbb{E}\left(e^{\lambda \xi_k f_k(t)}\right),$$

and because ξ_k are subnormal

$$\mathbb{E}(e^{\lambda P(t)}) \leq e^{\frac{\lambda^2 r}{2}}. \tag{80}$$

Since the measure of the whole space is finite we may suppose without restriction that $\mu(E) = 1$. Thus $\mu(I) \geq \frac{1}{\rho}$ and $P(t) \geq \frac{M}{2}$ or $-P(t) \geq \frac{M}{2}$ on the set I. Then using $1 \leq \rho\mu(I)$, $I \subset E$ and (80) we have

$$\begin{aligned} \mathbb{E}\left(e^{\lambda \frac{M}{2}}\right) &\leq \rho \mathbb{E}\left(\int_I e^{\lambda P(t)} + e^{-\lambda P(t)} d\mu(t) \right) \\ &\leq \rho \mathbb{E}\left(\int_E e^{\lambda P(t)} + e^{-\lambda P(t)} d\mu(t) \right) \leq 2\rho e^{\frac{\lambda^2 r}{2}}. \end{aligned} \tag{81}$$

Writing $2\rho e^{\frac{\lambda^2 r}{2}} = \frac{1}{\kappa} e^{\frac{\lambda^2 r}{2} + \log(2\rho\kappa)}$ in (81) gives

$$\mathbb{E}\left(e^{\frac{\lambda}{2}\left(M - \lambda r - \frac{2}{\lambda}\log(2\rho\kappa)\right)}\right) \leq \frac{1}{\kappa},$$

and thus

$$\mathcal{P}\left(M \geq \lambda r + \frac{2}{\lambda}\log(2\rho\kappa)\right) \leq \frac{1}{\kappa}.$$

Choosing $\lambda = (\log(2\rho\kappa))^{\frac{1}{2}}$ it follows that

$$\mathcal{P}\left(M \geq 3(r\log(2\rho\kappa))^{\frac{1}{2}}\right) \leq \frac{1}{\kappa}.$$

Now, if we choose $\rho = 2\pi N^2$ we get the claim if the interval satisfying (79) exists. Let us prove the following claim; If $p \in B$ is a trigonometric polynomial,

$$p(t) = \sum_{k=0}^{N} b_k \cos(kt + \phi_k), N \geq 2,$$

there exists an interval of length $\frac{1}{N^2}$, where $|p(t)| \geq \frac{1}{2}||p||_\infty$:
Making an estimate by taking the supremum norm of the integral in the definition of the Fourier coefficients b_k we have $b_k \leq \frac{4}{\pi}||p||_\infty$. On the other hand, calculating the derivative of $p(t)$,

$$p'(t) = -\sum_{k=1}^{N} k b_k \sin(kt + \phi_k),$$

we have

$$||p'||_\infty \leq \frac{2}{\pi} N(N+1)||p||_\infty \leq N^2||p||_\infty. \tag{82}$$

Now, since E is compact, there exists $t_0 > 0$ such that $||p||_\infty = \pm p(t_0)$. Therefore, by the mean value theorem

$$|p(t) - p(t_0)| \leq |t - t_0| \cdot ||p'||_\infty,$$

which implies that $|p(t)| \geq \frac{1}{2}||p||_\infty$ on the interval $[t_0 - \frac{1}{N^2}, t_0 + \frac{1}{N^2}]$. □

Lemma 4.5 also works on the n-dimensional torus with the definition of the dimension of the trigonometric polynomial

$$p(t_1, \ldots, t_n) = \sum c_{k_1,\ldots,k_n} e^{i(k_1 t_1 + \ldots + k_n t_n)}$$

given by $\sup(|k_1| + |k_2| + \ldots + |k_n|)$.

Proof. There is a cube defined on torus $\mathbb{T}^n$ satisfying condition (79) [9, p.70 Lemma]. $\square$

We can now prove that $W(t)$ is an a.s continuous function.

Theorem 4.6. *If s_j is a decreasing sequence and $\sum_{j=0}^{\infty} s_j < \infty$, then*

$$W(t) = X_0 t + \sum_{k=1}^{\infty} a_k \left[X_k \sin(2\pi k t) + Y_k (1 - \cos(2\pi k t)) \right]$$

represents a continuous function a.s.

The following proof is adapted from [9, p.84 Theorem 2].

Proof. First off, the function $W(t)$ converges if the functions $W_1(t), \ldots, W_n(t)$ converge. We are going to prove that the series

$$\sum_{k=1}^{\infty} a_k X_k^m \cos(kx + \phi_k)$$

converges uniformly a.s, where X_k^m is the m:th coordinate of the random vector X_k, $m = 1, \ldots, n$ and ϕ_k are some given numbers depending on k. In particular, we may choose $\phi_k = \frac{\pi}{2}$ to get the sine part of the series expansion of $W(t)$. Let $N_k = 2^{2^k}$ for $k \in \mathbb{N}$ and denote

$$P_k(x) = \sum_{N_k}^{N_{k+1}-1} a_k X_k^m \cos(kx + \phi_k).$$

By Lemma 4.5 we have

$$\mathcal{P}\left(||P_k||_\infty \geq C \left(\log N_{k+1} \sum_{N_k}^{N_{k+1}-1} (a_k)^2 \right)^{\frac{1}{2}} \right) \leq \frac{1}{N_{k+1}^2}, \tag{83}$$

where $C > 0$ is a constant. By the Borel-Cantelli lemma, we have that a.s

$$\begin{aligned} ||P_k||_\infty &= \mathcal{O}\left(\log(N_{k+1})^{\frac{1}{2}} \left(\sum_{N_k}^{N_{k+1}-1} (a_k)^2 \right)^{\frac{1}{2}} \right) \\ &= \mathcal{O}\left(2^{\frac{k}{2}} \left(\sum_{2^k \leq j < 2^{k+1}-1} (s_j)^2 \right)^{\frac{1}{2}} \right). \end{aligned}$$

We see that $\sum_{k=1}^{\infty} P_k(x)$ converges uniformly, if

$$\sum_{k=1}^{\infty} 2^{\frac{k}{2}} \left(\sum_{2^k \leq j < 2^{k+1}-1} (s_j)^2 \right)^{\frac{1}{2}} < \infty. \tag{84}$$

Therefore, if s_j is decreasing with

$$2^k s_{2^{k+1}} \leq 2^{\frac{k}{2}} \left(\sum_{2^k \leq j < 2^{k+1}-1} (s_j)^2 \right)^{\frac{1}{2}} \leq 2^k s_{2^k},$$

the condition (84) is equivalent to $\sum_{k=1}^{\infty} 2^k s_{2^k} < \infty$, which in turn is equivalent to $\sum_{k=1}^{\infty} s_k < \infty$, as was assumed. Now, $1 - \cos(kx) = 2\sin^2(\frac{kx}{2})$ so (83) does not change and the remaining proof stays the same. By choosing $x = 2\pi t$ and we have that each of the components $W_m(t)$ is a.s representing a continuous function as a sum of two functions with a.s uniformly convergent series. Thus $W(t)$ represents a.s a continuous function. $\square$

Next, we would like to see how mapping with a sample function of Brownian motion alters the Hausdorff dimension of a given set. For this we have the following theorem.

Theorem 4.7. *For a set $E \subset [0,1]$,*

$$\dim_{\mathrm{H}} W(E) = \inf\{n, 2\dim_{\mathrm{H}} E\} \quad a.s. \tag{85}$$

For now, let us postpone giving the proof of Theorem 4.7 since we are lacking some of the required results. With the Sections 4.2.2 and 4.2.3 in mind, we shall consider random trigonometric series of the form

$$F(t) = \sum_{k=0}^{\infty} a_k \left(X_k \cos(kt) + Y_k \sin(kt) \right), \tag{86}$$

where $X_0, Y_0, \ldots$ is a subnormal sequence. We may also assume that $a_k \geq 0$ since the series becomes similar if we replace a_k with $|a_k|$. Let us use the following notation:

$$\sigma = \liminf_{j\to\infty} \frac{-\log s_j}{j \log 2}, \tau = \limsup_{j\to\infty} \frac{-\log s_j}{j \log 2},$$

where s_j is defined like in (77). The term σ gives the upper limit for the exponent of Hölder continuity of the function F on the circle (see, [9, p.90]).

In addition, we need to consider capacity on a more general level. If $k(x)$ is a continuous positive valued function on $\mathbb{R}^n \setminus \{0\}$ with $\lim_{x\to 0} k(x) = \infty$, it is called a potential kernel. If $k(x)$ is a potential kernel with a positive valued Fourier transform $\widehat{k}$, it is said to be of positive type ([9, p.134]). Like before, the energy integral of a measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ with respect to potential kernel k is defined by

$$I_k(\mu) = \int |\hat{\mu}(\xi)|^2 \hat{k}(\xi) d\xi. \tag{87}$$

If $I_k(\mu) < \infty$ for some measure $\mu \in \mathcal{M}^1(A)$ for a compact set $A \subset \mathbb{R}^n$, then we say that the set A has positive capacity with respect to kernel k, like in the case with the Riesz kernel.
We are going to obtain Theorem 4.7 as a corollary from the following theorem.

Theorem 4.8. *Let $0 \le \sigma \le \tau \le 1$, and let function F be defined like* (86). *For a compact set E on the circle*

$$\inf\left\{n, \frac{1}{\tau}\dim_{\mathrm{H}} E\right\} \le \dim_{\mathrm{H}} F(E) \le \inf\left\{n, \frac{1}{\sigma}\dim_{\mathrm{H}} E\right\} \quad a.s. \tag{88}$$

The following proof is adapted from [9, p.200 Proposition 1, p.201 Theorem 1, Theorem 2, Theorem 3].

Proof. Because $F \in \Lambda^\alpha(\mathbb{T}^1, \mathbb{R}^n)$ for $0 \le \alpha < \sigma$ and E is compact set on the circle, the upper bound follows from Lemma 2.9. For the lower bound, let us proof the following statement: If k is a positive potential kernel on $\mathbb{R}^n$,

$$\kappa(t) = \int k(x) e^{-|x|^2/4\rho(t)} (\rho(t))^{-n/2} dx, \tag{89}$$

where $\rho(t) = \sum_{n=1}^\infty a_n^2(1-\cos(nt))$, and $\mathrm{Cap}_\kappa E > 0$, then $\mathrm{Cap}_k F(E) > 0$ a.s. In particular, if E has positive capacity with respect to $(\rho(t))^{-\alpha/2}, 0 < \alpha < n$, then $\mathrm{Cap}_\alpha F(E) > 0$ a.s:
Let θ be a Borel probability measure on the set E. Define a measure μ as a push-forward of θ, $\mu = F_*\theta$. By definition

$$\hat{\mu}(u) = \int e^{-i2\pi u\cdot\xi} d\mu(\xi) = \int_E e^{-i2\pi u\cdot F(t)} d\theta(t).$$

Next, let us calculate the expectation value of the $I_k(\mu)$. Using Fubini theorem twice we have

$$\begin{aligned}\mathbb{E}\left(I_k(\mu)\right) &= \mathbb{E}\left(\int \hat{k}(u)|\hat{\mu}(u)|^2 du\right) = \int \mathbb{E}\left(\hat{k}(u)|\hat{\mu}(u)|^2\right) du \\ &= \int \mathbb{E}\left(\hat{k}(u)\int_E\int_E e^{i2\pi u\cdot(F(t)-F(t'))}d\theta(t)d\theta(t')\right) du \\ &= \int_E\int_E\int \hat{k}(u)\mathbb{E}\left(e^{i2\pi u\cdot(F(t)-F(t'))}\right) dud\theta(t)d\theta(t').\end{aligned}$$

Let us calculate $\mathbb{E}\left(e^{i2\pi u\cdot(F(t)-F(t'))}\right)$. Note, that $F(t)$ can be written in the form $F(t)=\sum_{k=0}^{\infty} a_k \operatorname{Re}(Z_k e^{ikt})$, where $Z_k = X_k - iY_k$. Thus

$$\mathbb{E}\left(e^{i2\pi u\cdot(F(t)-F(t'))}\right) = \prod_{k=0}^{\infty}\mathbb{E}\left(e^{i2\pi u\cdot a_k \operatorname{Re}\left(Z_k e^{ikt}-Z_k e^{ikt'}\right)}\right). \tag{90}$$

Using Euler's formula, and since X_k and Y_k are subnormal, we have

$$\begin{aligned}&\mathbb{E}\left(e^{i2\pi u\cdot a_k \operatorname{Re}(Z_k e^{ikt}-Z_k e^{ikt'})}\right) \\ &= \mathbb{E}\left(e^{i2\pi u\cdot a_k X_k(\cos(kt)-\cos(kt'))}\right)\mathbb{E}\left(e^{i2\pi u\cdot a_k Y_k(\sin(kt)-\sin(kt'))}\right) \\ &= e^{-\pi|u|^2 a_k^2\left[(\cos^2(kt)-2\cos(kt)\cos(kt')+\cos^2(kt'))+\sin^2(kt)-2\sin(kt)\sin(kt')+\sin^2(kt')\right]} \\ &= e^{-\pi|u|^2 a_k^2[2-2\cos(kt)\cos(kt')-2\sin(kt)\sin(kt')]} \\ &= e^{-2\pi|u|^2 a_k^2\sin^2(k(t-t'))}.\end{aligned}$$

Thus (90) becomes

$$\prod_{k=0}^{\infty} e^{-2\pi|u|^2 a_k^2\sin^2(k(t-t'))} = e^{-|u|^2\sum_{k=0}^{\infty}2\pi a_k^2\sin^2(k(t-t'))} = e^{-|u|^2\rho(t-t')}.$$

Hencek, we have by Parseval's formula

$$\begin{aligned}\mathbb{E}(I_k(\mu)) &= \int_E\int_E\int \hat{k}(u)e^{-|u|^2\rho(t-t')}dud\theta(t)d\theta(t') \\ &= (2\pi)^{\frac{n}{2}}\int_E\int_E\int k(x)e^{-|x|^2/4\rho(t-t')}(2\rho(t-t'))^{-\frac{n}{2}}dxd\theta(t)d\theta(t') \\ &= \pi^{\frac{n}{2}}\int_E\int_E \kappa(t-t')d\theta(t)d\theta(t').\end{aligned}$$

If $\operatorname{Cap}_\kappa(E)>0$ we may choose a measure $\theta\not\equiv 0$ such that $\mathbb{E}(I_k(\mu))<\infty$, so μ has a.s finite energy with respect to k. Thus $\operatorname{Cap}_k F(E)>0$ a.s. The second part follows if we choose the kernel $k(x)=|x|^{-\alpha}$. Thus $\operatorname{Cap}_\alpha F(E)>0$ a.s if

$0 < \alpha < n$.
Next, if we estimate by $\rho(t) \geq \frac{1}{2}s_j^2$, when $\frac{\pi}{3}2^{-j} \leq t \leq \frac{2\pi}{3}2^{-j}$, we get that for $\tau' > \tau$,

$$\frac{1}{\rho(\tau)} = \mathcal{O}(|t|^{-2\tau'}),$$

as t tends to zero. Then $\mathrm{Cap}_{\alpha\tau'} E > 0$ and $\mathrm{Cap}_\alpha F(E) > 0$ by our calculation. But by the definition of the capacitary dimension, from $\dim_{\mathrm{H}} E > \alpha\tau$ it follows that $\mathrm{Cap}_{\alpha\tau'} E > 0$ for some $\tau' > \tau$, and from $\mathrm{Cap}_\alpha F(E) > 0$ it follows that $\dim_{\mathrm{H}} F(E) \geq \alpha$. Thus a.s

$$\dim_{\mathrm{H}} F(E) \geq \frac{1}{\tau} \dim_{\mathrm{H}} E,$$

which concludes the proof. □

We can now prove Theorem 4.7 with some notes from [9, p.203].

Proof. If we restrict ourselves on a line instead of the circle and pick

$$\rho(t) = \frac{1}{2}a_0^2 t^2 + \sum_{k=1}^{\infty} a_k^2 (1 - \cos(kt)),$$

we get the result for $W(t)$ from Theorem 4.8. This can be seen by repeating the calculation following equation (90) with function $W(t)$ instead of $F(t)$. Now we need to calculate the value of σ and τ. Let us estimate the value of s_j. By definition

$$s_j = \left(2 \cdot \sum_{2^j \leq k < 2^{j+1}} a_k^2\right)^{\frac{1}{2}} = \left(2C \cdot \sum_{2^j \leq k < 2^{j+1}} k^{-2}\right)^{\frac{1}{2}}.$$

Now we get the upper bound

$$s_j \leq \left(C \cdot 2^{-2j} 2^j\right)^{\frac{1}{2}} = C' \cdot 2^{-\frac{1}{2}j}, \tag{91}$$

and the lower bound

$$s_j \geq \left(C \cdot 2^{-2(j+1)} 2^j\right)^{\frac{1}{2}} = C'' \cdot 2^{-\frac{1}{2}j}. \tag{92}$$

Using the estimate (91) we get

$$\sigma = \liminf_{j\to\infty} \frac{-\log s_j}{j \log 2} \geq \liminf_{j\to\infty} \frac{j \log 2 + C}{2j \log 2} = \frac{1}{2},$$

and by using the estimate (92) we have

$$\tau = \limsup_{j\to\infty} \frac{-\log s_j}{j\log 2} \leq \limsup_{j\to\infty} \frac{j\log 2 + C'}{2j\log 2} = \frac{1}{2}.$$

Thus $\sigma = \tau = \frac{1}{2}$ for $W(t)$ and we finally get, that a.s

$$\inf\{n, 2\dim_{\mathrm{H}} E\} \leq \dim_{\mathrm{H}} W(E) \leq \inf\{n, 2\dim_{\mathrm{H}} E\}.$$

□

We now know the Hausdorff dimension of the Brownian image of a compact set from a line. The final step would be to construct a measure with support on the image, with the Fourier dimension bound from below by the Hausdorff dimension of the image given by Theorem 4.7. We do this in the proof of the following theorem.

Theorem 4.9. *Let $E \subset \mathbb{R}^n$ be a compact set on a line with $\dim_{\mathrm{H}} E = \alpha < \frac{n}{2}$. Then $W(E)$ is a.s a Salem set of dimension 2α.*

Let us first consider some lemmas. These results will be used quite often through the rest of the study.

Lemma 4.10. *Denote by Q the interior of the unit cube $Q_n \subset \mathbb{R}^n$. Let $E \subset Q$ be a compact set, $\mu \in \mathcal{M}(E)$ a measure and $\phi(t), \psi(t)$ positive decreasing functions of $t > 0$ such that*

$$\phi\left(\frac{1}{2}t\right) = \mathcal{O}(\phi(t)),\ \ \psi\left(\frac{1}{2}t\right) = \mathcal{O}\left(\psi(t)\right), \quad (t \to \infty).$$

If $\hat{\mu}(k) = \mathcal{O}\left(\phi(|k|)/\psi(|k|)\right)$, $k = (k_1, \ldots, k_n) \to \infty$, k_j are integers, then

$$\hat{\mu}(u) = \mathcal{O}\left(\phi(|u|)/\psi(|u|)\right), \quad u = (u_1, \ldots, u_n) \to \infty.$$

The following proof is adapted from [9, p.252 Lemma 1].

Proof. Let $\gamma \in C^\infty$, with support in a compact set in Q, such that $\gamma(x) = 1$ for $x \in \operatorname{spt}\mu$. For every $a, x \in Q_n$ let

$$\gamma_a(x) = e^{i2\pi a\cdot x}\gamma(x) = \sum_{k\in\mathbb{Z}^n} \hat{\gamma}_a(k) e^{i2\pi k\cdot x}. \tag{93}$$

Because the derivatives of γ_a are uniformly bounded with respect to a, for all $q > 0$, $|\hat{\gamma}_a(n)| \leq C|n|^{-q}$, where the constant C depends only on γ and q.

Next, let us consider $\hat{\mu}(a+m)$, where $m \in \mathbb{Z}^n$, which can be written with (93) as

$$\begin{aligned}\hat{\mu}(a+m) &= \int e^{i2\pi a\cdot x} e^{i2\pi m\cdot x} d\mu(x) = \int \gamma_a(x) e^{i2\pi m\cdot x} d\mu(x) \\ &= \sum_{k\in\mathbb{Z}^n} \hat{\gamma}_a(k)\hat{\mu}(k+m). \end{aligned} \tag{94}$$

We want to estimate the norm of (94), for which let us divide the consideration into two parts. The first is, when $|k| \le \frac{1}{2}|m|$ and the second, when $|k| > \frac{1}{2}|m|$. By assumption on the measure μ we have

$$|\hat{\mu}(k)| < \phi(|k|)/\psi(|k|)$$

and we may suppose $\kappa > 0$ is a constant such that $|\hat{\mu}(k)| < \kappa$ for all k. The latter can be found since $\hat{\mu}(0) = \mu(E) < \infty$. In addition, by assumptions on functions ϕ and ψ we have

$$\sum_{|k|\le\frac{1}{2}|m|} \hat{\gamma}_a(k)\hat{\mu}(k+m) \le \left(\phi(\frac{1}{2}|m|)/\psi(2|m|)\right) \sum_{k\in\mathbb{Z}^n} |\hat{\gamma}_a(k)|, \tag{95}$$

and

$$\sum_{|k|>\frac{1}{2}|m|} \hat{\gamma}_a(k)\hat{\mu}(k+m) \le \kappa \sum_{|k|>\frac{1}{2}|m|} |\hat{\gamma}_a(k)|. \tag{96}$$

Therefore, combining (95) and (96) with preliminaries on functions of $\mathcal{S}(\mathbb{R}^n)$,

$$|\hat{\mu}(k+m)| \le C_1 \left(\phi(\frac{1}{2}|m|)/\psi(2|m|)\right) + C_2|m|^{-r},$$

where $r > 0$ is arbitrary and C_1, C_2 are constants independent of m. Also, by assumptions on ϕ and ψ, for some $r > 0$,

$$\frac{\phi(t)}{\psi(t)} > \frac{\phi(0)}{\psi(0)} t^{-r},$$

so $|\hat{\mu}(k+m)| \le C(\phi(|m|)/\psi(|m|))$ and thus, $\hat{\mu}(u) = \mathcal{O}(\phi(|u|)/\psi(|u|))$. $\square$

Next, let us prove the following variation of Frostman's lemma.

Lemma 4.11. *Let $h(t)$ be a strictly increasing, continuous and positive valued function of $t > 0$ with $h(0) = 0$, $h(2t) = \mathcal{O}(h(t))$. If $E \subset \mathbb{R}^n$ is a compact set on a line with positive h-measure, then E supports a measure θ such that $\theta(I) \le Ch(|I|)$ for all intervals I. Here C is a constant only depending on θ.*

The following proof is adapted from [10, Theorem 2.7] with notes from [9, Chapter 10].

Proof. Let $\theta \in \mathcal{M}(E)$ be a measure such that $\theta(I) \leq Ch(|I|)$ for all intervals I. If E can be covered with intervals $I_1, I_2, \ldots$, then

$$0 < \theta(E) \leq \sum_i \theta(I_i) \leq C \sum_i h(|I_i|).$$

Thus $\mathcal{H}^{\mathrm{h}}(E) > 0$. Next, the other direction. Because E is a compact set on a line, by translation, we may assume that E is contained in some dyadic line $\mathcal{I}_0 \subset \mathbb{R}^n$. That is, a line starting from the origin of length 2^N for some $N \in \mathbb{N}$. Because $\mathcal{H}^{\mathrm{h}}(E) > 0$, also $\mathcal{H}^{\mathrm{h}}_\infty(E) > 0$ and thus there exists a constant $c > 0$ such that, for $b := c\,\mathcal{H}^{\mathrm{h}}_\infty(E)$,

$$\sum_i h(I_i) \geq b, \tag{97}$$

whenever intervals $I_1, I_2, \ldots$ cover the set E. For $m = 1, 2, \ldots$, let $\mathcal{I}_m$ be the set of all dyadic intervals of length 2^{-m} of the line $\mathcal{I}_0$. Define a measure θ^m_m on $\mathbb{R}^n$ by requiring for all $\mathcal{I} \in \mathcal{I}_m$,

$$\theta^m_{m|\mathcal{I}} = \begin{cases} h(2^{-m})\, \mathcal{L}^{\mathrm{n}}(\mathcal{I})^{-1}\, \mathcal{L}^{\mathrm{n}}{}_{|\mathcal{I}}, & \text{if } E \cap \mathcal{I} \neq \emptyset \\ 0, & \text{if } E \cap \mathcal{I} = \emptyset. \end{cases}$$

Next, transform θ^m_m into a measure θ^m_{m-1} by requiring for all $\mathcal{I} \in \mathcal{I}_{m-1}$

$$\theta^m_{m-1|\mathcal{I}} = \begin{cases} h(2^{-(m-1)})\theta^m_m(\mathcal{I})^{-1}\theta^m_{m|\mathcal{I}}, & \text{if } \theta^m_m(\mathcal{I}) > h(2^{-(m-1)}) \\ \theta^m_m, & \text{if } \theta^m_m(\mathcal{I}) \leq h(2^{-(m-1)}). \end{cases}$$

Continuing this way, θ^m_{m-k-1} is given by θ^m_{m-k} by requiring for all $\mathcal{I} \in \mathcal{I}_{m-k-1}$

$$\theta^m_{m-k-1|\mathcal{I}} = \begin{cases} h(2^{-(m-k-1)})\theta^m_{m-k}(\mathcal{I})^{-1}\theta^m_{m-k|\mathcal{I}}, & \text{if } \theta^m_m(\mathcal{I}) > h(2^{-(m-k-1)}) \\ \theta^m_{m-k}, & \text{if } \theta^m_{m-k}(\mathcal{I}) \leq h(2^{-(m-k-1)}). \end{cases}$$

The process ends when $E \subset \mathcal{I}$ for some $\mathcal{I} \in \mathcal{I}_{m-k_0}$ and $\theta^m = \theta^m_{m-k_0}$. Furthermore, none of the intervals gains more measure during the process so $\theta^m(\mathcal{I}) \leq h(2^{-(m-k)})$ for $\mathcal{I} \in \mathcal{I}_{m-k}, k = 0, 1, \ldots$ In particular, for every $x \in E$ there exists k and $\mathcal{I} \in \mathcal{I}_{m-k}$ such that if $x \in \mathcal{I}$, then $\theta^m(\mathcal{I}) = h(|\mathcal{I}|)$. Choosing the biggest such interval $\mathcal{I}$ for every $x \in E$, we get disjoint intervals $I_1, I_2, \ldots$ such that

$$E \subset \bigcup_{i=1}^k I_i \quad \text{and} \quad \theta^m(\mathbb{R}^n) = \sum_{i=1}^k \theta^m(I_i) = \sum_{i=1}^k h(|I_i|) \geq b,$$

where last inequality is due to (97). Setting $\mu^m = \theta^m(\mathbb{R}^n)^{-1}\theta^m$ we have $\mu^m(\mathbb{R}^n) = 1$ and $\mu^m(\mathcal{I}) \leq b^{-1}h(|\mathcal{I}|)$ for all $\mathcal{I} \in \mathcal{I}_{m-k}, k = 0, 1, \ldots$.
Now μ^m converges weakly into a measure θ, $\theta \in \mathcal{M}(E)$ and $\theta(E) = 1$. In addition, $\mu^m(I) \leq b^{-1}h(|I|)$ for all intervals I, so $\theta(I) \leq b^{-1}h(|I|)$ for all intervals I. □

For a compact set that is not contained on a line, we may replace the dyadic lines used in Lemma 4.11 with balls or dyadic cubes and get the result (see [9, p.132]). We are going to apply Lemma 4.11 to either concave or convex functions. The following lemma is just a calculation using properties of positive concave functions. Turns out that it is a really handy way of getting estimates in cases we consider.

Lemma 4.12. *Let $h(t)$ be a positive concave function of $t > 0$. For $y > 0$, define a function*

$$h^*(y) = \int_0^\infty e^{-t}h(ty^{-1})dt$$

Then $h^*(y) = \mathcal{O}(h(y^{-1}))$ as $y \to \infty$.

Proof. Let $y > 0$. Since the function h is concave, for $0 \leq t \leq 1$ we have $h(ty^{-1}) \leq h(y^{-1})$, and for $t > 1$ we have $h(ty^{-1}) \leq th(y^{-1})$. Therefore

$$\begin{aligned}
h^*(y) &= \int_0^\infty e^{-t}h(ty^{-1})dt \\
&= \int_0^1 e^{-t}h(ty^{-1})dt + \int_1^\infty e^{-t}h(ty^{-1})dt \\
&\leq \int_0^1 e^{-t}h(y^{-1})dt + \int_1^\infty e^{-t}th(y^{-1})dt \\
&= \left(\int_0^1 e^{-t}dt + \int_1^\infty e^{-t}tdt\right) h(y^{-1}) = Ch(y^{-1}),
\end{aligned}$$

proving the estimate. □

Now we have all the results needed to prove Theorem 4.9. The following proof is adapted from [9, p.251 Theorem 1].

Proof. Let $h(t)$ be a positive concave function of $t > 0$ and θ a probability measure on the positive half-line such that $\theta(I) \leq h(|I|)$ for all intervals I. Write the push-forward measure of θ under the Wiener function W as $\mu = W_*\theta$, with the Fourier transform given by

$$\hat{\mu}(u) = \int_0^\infty e^{-i2\pi u \cdot W(t)}d\theta(t).$$

Next, let us estimate $|\hat{\mu}(u)|$. For an integer $q \geq 1$, we consider

$$|\hat{\mu}(u)|^{2q} = \int_0^\infty \cdots \int_0^\infty e^{-i2\pi u\cdot\left[W(s_1)+\ldots+W(s_q)+W(s_1')+\ldots+W(s_q')\right]}\times$$
$$\times d\theta(s_1)\cdots d\theta(s_q)d\theta(s_1')\cdots d\theta(s_q')$$
$$= (q!)^2 \int_{0\leq s_1\leq\ldots\leq s_q} \int_{0\leq s_1'\leq\ldots\leq s_q'} e^{-i2\pi u\cdot[\cdots]} d\theta(s_1)\cdots d\theta(s_q'). \qquad (98)$$

The last equality is due to Fubini as the integrand is symmetric with respect to $s_1,\ldots,s_q$ and with respect to $s_1',\ldots,s_q'$. Next, fix a sequence $s_1,\ldots,s_q,s_1',\ldots,s_q'$ and order it as an increasing sequence $t_1,\ldots,t_{2q}$. Denoting by $\varepsilon_j = \pm 1$ for $j = 1,\ldots,2q$, we get systems $\{\varepsilon_j\}$ for which

$$\varepsilon_1 + \ldots + \varepsilon_{2q} = 0,$$

and the integral (98) becomes

$$|\hat{\mu}(u)|^{2q} = (q!)^2 \sum_{\varepsilon_j} \int_{0\leq t_1\leq\ldots\leq t_{2q}} e^{-i2\pi u[\varepsilon_1 W(t_1)+\varepsilon_2 W(t_2)+\ldots\varepsilon_{2q}W(t_{2q})]}\times$$
$$\times d\theta(t_1)\cdots d\theta(t_{2q}). \qquad (99)$$

Because $0 \leq t_1 \leq \ldots \leq t_{2q}$, $W(t_1), W(t_2)-W(t_1),\ldots,W(t_{2q})-W(t_{2q-1})$ are independent Gaussian vectors. Thus

$$\mathbb{E}\left(e^{-i2\pi u\cdot[\varepsilon_1 W(t_1)+\varepsilon_2 W(t_2)+\ldots\varepsilon_{2q}W(t_{2q})]}\right)$$
$$= \mathbb{E}\left(e^{-i2\pi u\cdot[(\varepsilon_1+\ldots+\varepsilon_{2q})W(t_1)+(\varepsilon_2+\ldots+\varepsilon_{2q})(W(t_2)-W(t_1))+\ldots+\varepsilon_{2q}(W(t_{2q})-W(t_{2q-1}))]}\right)$$
$$= e^{-\pi|u|^2\left[t_1(\varepsilon_1+\ldots+\varepsilon_{2q})^2+(t_2-t_1)(\varepsilon_2+\ldots+\varepsilon_{2q})^2+\ldots+(t_{2q}-t_{2q-1})\varepsilon_{2q}^2\right]}. \qquad (100)$$

Denoting by $\psi_j = \pi|u|^2(\varepsilon_j+\ldots+\varepsilon_{2q})^2$, we have $\psi_j \geq 0$ for all j and $\psi_j \geq \pi|u|^2$ for even j. Therefore, applying Fubini we get from (99) with use of (100) that

$$\mathbb{E}(|\hat{\mu}(u)|^{2q}) = (q!)^2 \sum_{\varepsilon_j} \int_{0\leq t_1\leq\ldots\leq t_{2q}} e^{-[t_1\psi_1+(t_2-t_1)\psi_2+\ldots+(t_{2q}-t_{2q-1})\psi_{2q}]}\times$$
$$\times d\theta(t_1)\cdots d\theta(t_{2q}). \quad (101)$$

Integrating over each even j, that is, $j = 2l$, $l = 1,2,\ldots,q$ we have

$$\int_{t_{2l-1}}^{t_{2l}} e^{-(t_{2l}-t_{2l-1})\psi_{2l}} d\theta(t_{2l}-t_{2l-1}) \leq \int_0^\infty e^{-\pi|u|^2 t} d\theta(t-t_{2l-1})$$
$$\leq \pi|u|^2 \int_0^\infty e^{-\pi|u|^2 t} h(t)dt = h^*\left(\pi|u|^2\right),$$

where second inequality is due to $\theta(t) \leq h(t)$. For other values of j, we use estimate $|e^{i2\pi\xi}| \leq 1$. There are $q!$ ways to order the remaining q integrals and the measure of the whole space is 1. By Fubini theorem, all the integrals have the same value, and hence we have that integral over one such ordering is

$$\int_{0\leq t_1\leq t_3\leq\ldots\leq t_{2q-1}} 1 d\theta(t_1)d\theta(t_3)\cdots d\theta(t_{2q-1}) = \frac{1}{q!}.$$

Lastly, there are in total $\binom{2q}{q} = (2q)!/(q!)^2$ choices for systems $\{\varepsilon_j\}$. Thus, for (101) we have

$$\begin{aligned}
\mathbb{E}(|\hat{\mu}(u)|^{2q}) &\leq (q!)^2 \sum_{\varepsilon_j} \left(h^*\left(\pi|u|^2\right)\right)^q \int_{0\leq t_1\leq\ldots\leq t_{2q}} d\theta(t_1)\cdots d\theta(t_{2q}) \\
&= \frac{(2q)!}{q!}\left(h^*\left(\pi|u|^2\right)\right)^q,
\end{aligned}$$

and by Lemma 4.12

$$\mathbb{E}(|\hat{\mu}(u)|^{2q}) \leq \left(Cqh\left(|u|^{-2}\right)\right)^q. \tag{102}$$

Writing (102) for all $u = k \in \mathbb{Z}^n$ such that $q = q_k = [\log|k|]$ we get further estimate

$$\begin{aligned}
\mathbb{E}\left(\sum_{k\in\mathbb{Z}^n,k\neq 0} |k|^{-n-1}\left(\frac{|\hat{\mu}(k)|^2}{Cqh(|k|^{-2})}\right)^{q_k}\right) &\leq \sum_{k\in\mathbb{Z}^n,k\neq 0} |k|^{-n-1} \\
&\leq \int_{\mathbb{R}^n} |x|^{-n-1}dx < \infty.
\end{aligned}$$

Hence the general term of the above series

$$|k|^{-n-1}\left(\frac{|\hat{\mu}(k)|^2}{Cqh(|k|^{-2})}\right)^{q_k}$$

tends to zero a.s as $|k| \to \infty$, and therefore a.s

$$|\hat{\mu}(k)|^2 \leq c|k|^{n+1/q_k}h(|k|^{-2}) \leq C\log(|k|)h(|k|^{-2}).$$

This shows that a.s $\hat{\mu}(k) = \mathcal{O}\left(\sqrt{\log(|k|)h(|k|^{-2})}\right)$. By repeating the above calculation for given $\varepsilon > 0$, we have that a.s

$$\hat{\mu}(\varepsilon k) = \mathcal{O}\left(\sqrt{\log(|\varepsilon k|)h(|\varepsilon k|^{-2})}\right). \tag{103}$$

If (103) holds and μ is supported by a compact set of diameter less than $\frac{1}{\varepsilon}$, we may apply Lemma 4.10 to a measure μ_ε such that $\hat{\mu}_\varepsilon(u) = \hat{\mu}(\varepsilon u)$, i.e μ_ε has compact support in the unit cube, we obtain that a.s

$$\hat{\mu}(u) = \mathcal{O}(\sqrt{\log(|u|)h(|u|^{-2})}). \tag{104}$$

We have shown that under the assumptions of Lemma 4.11 we get the asymptotic estimate (104). If we choose a concave function

$$h(t) = \log(1 + t^\alpha)$$

and use the estimate $\log(1 + x) \leq x$ for all $x > -1$, it follows that

$$h\left(|u|^{-2}\right) \leq |u|^{-2\alpha}.$$

Also, by noting that $\log|u| \leq |u|^{2\varepsilon}$ for $0 \leq |u| \leq 1$ and any $\varepsilon > 0$, from (104) we get that $W(E)$ carries a.s a measure $\mu \not\equiv 0$ such that its Fourier transform satisfies

$$\hat{\mu}(u) = \mathcal{O}(|u|^{-\alpha-\varepsilon})$$

for each $\varepsilon > 0$. Therefore, $\dim_{\mathrm{F}} W(E) = 2\alpha$ a.s. Combining this with (85) we finally get that $W(E)$ is a.s a Salem set. $\square$

4.2.2 Images of sets and measures under Gaussian Fourier series

In this section we aim to generalize the result of Theorem 4.9. We consider n-dimensional Gaussian Fourier series,

$$\sum_{k=0}^{\infty} a_k \left(X_k \cos(kt) + Y_k \sin(kt)\right) \tag{105}$$

where for all k the coefficients $a_k \geq 0$ and X_k, Y_k are independent Gaussian random variables with

$$\mathbb{E}(e^{i2\pi u \cdot X_k}) = \mathbb{E}(e^{i2\pi u \cdot Y_k}) = e^{-\pi|u|^2}.$$

If the series (105) defines a continuous function, we denote it by $F(t)$. To include the case of Brownian motion, we may consider the series

$$\sum_{k=1,\, k\,\mathrm{odd}}^{\infty} a_k \left(X_k \cos(kt) + Y_k \sin(kt)\right). \tag{106}$$

Like before, $h(t)$ is a positive concave function on the positive half-line, θ denotes a probability measure supported by the circle and the measure μ is defined as a push-forward of F, $\mu = F_*\theta$.

Theorem 4.13. *Suppose that $\theta(I) \le h(|I|)$ for all the intervals I and let $a_k \ge k^{-\frac{1}{2}-\beta}$, $\beta > 0$, for series defined as (105) or (106). Then a.s*

$$\hat{\mu}(u) = \mathcal{O}\left(\sqrt{\log|u| h(|u|^{-1/\beta})}\right).$$

The following proof is adapted from [9, p.258, Lemma 6, Theorem 3].

Proof. We would like to estimate $\mathbb{E}(|\hat{\mu}(u)|^{2q})$ like in the proof of Theorem 4.9. For that we write $F(t) = \sum_{k=0}^{\infty} a_k \operatorname{Re}(Z_k e^{ikt})$, where $Z_k = X_k - iY_k$. With the same calculation as in the proof of Theorem 4.8 we have

$$\mathbb{E}\Big(e^{i2\pi u\cdot(F(t_1)+\ldots+F(t_q)-F(s_1)-\ldots-F(s_q))}\Big) = e^{\pi|u|^2\sum_{k=0}^{\infty} a_k^2 |e^{ikt_1}+\ldots+e^{ikt_q}-e^{iks_1}-\ldots-e^{iks_q}|^2}.$$

For clearer representation, let us denote by

$$\delta(kt, ks) = |e^{ikt_1} + \ldots + e^{ikt_q} - e^{iks_1} - \ldots - e^{iks_q}|^2$$

so we get that

$$\mathbb{E}\left(e^{i2\pi u\cdot(F(t_1)+\ldots+F(t_q)-F(s_1)-\ldots-F(s_q))}\right) = e^{-\pi|u|^2\sum_k a_k^2\delta(kt,ks)}.$$

Thus the expectation value we are after becomes

$$\mathbb{E}\left(|\hat{\mu}(u)|^{2q}\right) = \int\cdots\int e^{-\pi|u|^2\sum_k a_k^2\delta(kt,ks)} d\theta(t_1)\ldots d\theta(t_q) d\theta(s_1)\ldots d\theta(s_q). \tag{107}$$

Next, to estimate the integrand in (107), let us prove the following statement: Suppose that $\theta(I) \le h(|I|)$ for all intervals I. For a given integer N, define a function Δ on the $2q$-dimensional torus $\mathbb{T}^{2q}$ as

$$\Delta(t, s) = \sum_{k=-N}^{N} \delta(kt, ks),$$

and let $\theta \times \ldots \times \theta$ be a product measure on $\mathbb{T}^{2q}$. Then $\Delta(t, s) > cN$, where $c > 0$ is a constant, outside of an exceptional set G with

$$\theta \times \ldots \times \theta(G) \le (40qh(1/N))^q;$$

If $t = (t_1, \ldots, t_q)$ is given on $\mathbb{T}^q$ and $\varepsilon > 0$, we are going to define a set $F(t, \varepsilon)$ on the circle satisfying

a) $\theta(F(t, \varepsilon)) \le 4qh(\varepsilon)$

b) $\sum_{k=1}^{q} |s - t_k|^{-2} < \frac{\pi^2}{6\varepsilon^2}, \quad s \notin F(t, \varepsilon)$.

The following is known as a construction of H. Cartan: For all $j \leq q$, choose all the intervals of length $2\varepsilon j$, which contain at least j points t_k. Let I_1 be the interval with the largest number j and remove the points t_k. Repeat, and we get the possibly empty interval I_2. By removing the points $t_k \in I_1 \cup I_2$, we get the interval I_3 and so on. Let J_l be the interval which is obtained by doubling the length of interval I_l with respect to the center point. Then

$$F(t, \varepsilon) = \bigcup_l J_l :$$

Since at the end of the construction we have at most q intervals of length 4ε, $F(t, q)$ is contained in a union of $4q$ intervals of length ε. Thus condition a) is satisfied. If $s \notin \bigcup_l J_l$, the numbers t_k can be reordered such that

$$|s - t_1| \leq |s - t_2| \leq \ldots \leq |s - t_q|.$$

From condition $s \notin \bigcup_l J_l$ it follows that point s can not be $|\frac{1}{2} I_l|$ distance further from any t_k for any l. Then, if for some j we had $|s - t_j| < \varepsilon j$, none of the points $t_1, \ldots, t_q$ would belong to interval I_l of length greater-equal to $2\varepsilon j$. On the other hand, then $\{t_1, \ldots, t_q\} \subset [s - j\varepsilon, s + j\varepsilon]$, which is a contradiction with the construction. Thus

$$|s - t_1| \geq \varepsilon, |s - t_2| \geq 2\varepsilon, \ldots, |s - t_q| \geq q\varepsilon,$$

giving condition b). Next, let us define the exceptional set as

$$G = \{(t, s) : s_j \in F(t, \varepsilon), j = 1, 2, \ldots, q\}. \tag{108}$$

Then by a), $\theta \times \ldots \times \theta(G) \leq (4qh(\varepsilon))^q$. Let us assume that $(t, s) \notin G$, that is $s_j \notin F(t, \varepsilon)$ for some j. Define a function

$$\gamma(t) := \sum_{k=-N}^{N} \hat{\gamma}_k e^{ikt} = K_N(t - s_j),$$

where K_N denotes the N:th Fejér kernel, $K_N(x) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{s=-k}^{k} e^{isx}$. Then for any $j = 1, \ldots, q$ we have

$$\gamma(s_j) - \gamma(t_1) - \ldots - \gamma(t_q) \geq N - \frac{\pi^2}{N} \sum_{k=1}^{q} (s_j - t_k)^{-2} \geq N - \frac{\pi^4}{6N\varepsilon^2}. \tag{109}$$

By the Cauchy-Schwarz inequality we have

$$\gamma(s_1) + \ldots + \gamma(s_q) - \gamma(t_1) - \ldots - \gamma(t_q) \leq \left(\sum_{k=-N}^{N} |\hat{\gamma}_k|^2 \right)^{\frac{1}{2}} (\Delta(t, s))^{\frac{1}{2}}, \tag{110}$$

and hence rearraging (110) combined with (109) we have

$$\Delta(t,s) \geq \left(N - \frac{\pi^4}{6N\varepsilon^2}\right)^2 \left(\sum_{k=-N}^{N} |\hat{\gamma}_k|^2\right)^{-1}, \tag{111}$$

and by choosing $\varepsilon = \pi^2/N$ in (111), and noting that $|\hat{\gamma}_k| \leq 1$ for each k, we get that $\Delta(t,s) > cN$, where $c > 0$ is a constant. In addition,

$$(4qh(\varepsilon))^q \leq (4\pi^2 qh(1/N))^q \leq (40qh(1/N))^q,$$

proving our statement.
Now, let us divide (107) to sums of integrals over sets $G_{v-1} \setminus G_v$, where $G_0 = \mathbb{T}^{2q}$ and G_v is the exceptional set corresponding to $N_v = 2^v$, defined in (108). In case of series (105)

$$\mathbb{E}\left(|\hat{\mu}(u)|^{2q}\right) \leq \sum_{v=1}^{\infty} \left(Cqh(2^{-v})\right)^q e^{-\pi|u|^2 2^v \eta_v}, \tag{112}$$

where $C > 0$ is a constant and $\eta_v = \inf_{k \leq 2^v} a_k^2$. Since $a_k \geq k^{-\frac{1}{2}-\beta}$ we have $2^v \eta_v \geq 2^{-2\beta v}$. When $2^{\beta\xi} < |u| < e^{\beta(\xi+1)}$, we write (112) in two parts, firstly $1 \leq v \leq \xi$, and secondly $\xi + 1 \leq v \leq \infty$. Since the function h is concave, in the first part we have $h(2^{-v}) \leq 2^{\xi - v} h(2^{-\xi})$ and in the second part $h(2^{-v}) \leq h(2^{-\xi})$. Hence for (112)

$$\mathbb{E}\left(|\hat{\mu}(u)|^{2q}\right) \leq \left(\tilde{C}qh(2^{-v})\right)^q + \left(\tilde{C}'qh(2^{-v})\right)^q \leq \left(Cqh(|u|^{-1/\beta})\right)^q.$$

Then a.s

$$\hat{\mu}(u) = \mathcal{O}\left(\sqrt{\log(|u|)h(|u|^{-1/\beta})}\right)$$

by Lemma 4.10 with the same arguments as in the proof of Theorem 4.9. For series (106), the result follows with the same proof, only replacing the function $\Delta(t,s)$ with function $\Delta(2t,2s)$. $\square$

Now, we have all the results needed to generalize Theorem 4.9. It comes in the form of the following theorem.

Theorem 4.14. *Let $F(t)$ be defined by (105) or by (106) with coefficients $a_k = k^{-\frac{1}{2}-\beta}$, where $\beta > 0$. If $\beta \leq 1$ and E is a compact set on the circle with $\dim_{\mathrm{H}} E = \alpha < n\beta$, then $F(E)$ is a.s a Salem set with dimension α/β.*

Proof. Suppose the assumptions of the theorem are satisfied. First,

$$s_j = \left(2 \cdot \sum_{2^j \leq k < 2^{j+1}} a_k^2\right)^{\frac{1}{2}} = \left(2 \cdot \sum_{2^j \leq k < 2^{j+1}} \left(k^{-\frac{1}{2}-\beta}\right)^2\right)^{\frac{1}{2}}.$$

Now, an upper bound for s_j is given by

$$s_j \leq \left(2 \cdot 2^{-(1+2\beta)j} 2^j\right)^{\frac{1}{2}} = 2^{\frac{1}{2}} \cdot 2^{-\beta j}, \tag{113}$$

and hence by (113) we get

$$\sigma = \liminf_{j\to\infty} \frac{-\log s_j}{j \log 2} \geq \liminf_{j\to\infty} \frac{\beta j \log 2 + C}{j \log 2} = \beta,$$

Thus by Theorem 4.8 we have that a.s

$$\dim_{\mathrm{H}} F(E) \leq \inf\left\{n, \frac{1}{\beta}\dim_{\mathrm{H}} E\right\} = \frac{\alpha}{\beta}. \tag{114}$$

Under the assumptions of Theorem 4.13 E supports a.s a measure for which

$$\hat{\mu}(u) = \mathcal{O}\left(\sqrt{\log(|u|)h(|u|^{-1/\beta})}\right).$$

If we choose a concave function

$$h(t) = \log(1 + t^{\alpha})$$

and estimate logarithm like in the proof of Theorem 4.9, we have by Lemma 4.11 that $F(E)$ supports a.s a measure $\mu \not\equiv 0$ for which

$$\hat{\mu}(u) = \mathcal{O}\left(|u|^{-\frac{\alpha}{2\beta}+\frac{\varepsilon}{2}}\right) \tag{115}$$

for all $\varepsilon > 0$. Then by (115), combined with (114), we have that a.s

$$\dim_{\mathrm{F}} F(E) \geq \frac{\alpha}{\beta} = \dim_{\mathrm{H}} F(E).$$

Therefore $F(E)$ is a.s a Salem set of dimension α/β. $\square$

4.2.3 Images of sets and measures under fractional Brownian motion

In this section, we focus on the images of sets and measures under fractional Brownian motion. Fractional Brownian motion is the generalization of the Wiener process and like in Section 4.2.1 we aim to find more random Salem sets. Since most of the required facts concerning the continuity properties of (n, d, γ) were given in the preliminaries we may prove the following theorem relying on them.

Theorem 4.15. *Let $X(t)$ be an a.s continuous version of (n, d, γ)-Gaussian process and $h(t)$ either a positive convex or concave function of $t > 0$ with $h(2t) = \mathcal{O}(h(t))$, $(t \to 0)$. If $E \subset \mathbb{R}^n$ is a compact set such that $\mathcal{H}^{\mathrm{h}}(E) > 0$ then $X(E)$ supports a.s a measure $\mu \not\equiv 0$ such that*

$$\hat{\mu}(\xi) = \mathcal{O}\left(\sqrt{h(|\xi|^{-1/\beta})\log|\xi|}\right), \ \ as\ |\xi| \to \infty.$$

Additionally, if $\dim_{\mathrm{H}} E = \alpha < n\beta$, then $X(E)$ is a.s a Salem set of dimension α/β.

The following proof is adapted from [9, Chapter 18, Sections 1-3].

Proof. Let $E \subset \mathbb{R}^n$ be a compact set and $X(t)$ an a.s continuous version of a (n, d, γ)-process. Because E is compact we may, by dilation, assume without lost of generality that $\operatorname{diam} E \leq 1$. Let $t, s \in E$. Since the process $X(t)$ has a.s a modulus of continuity $\omega_X(h) = \mathcal{O}\left(\sqrt{|h|^\gamma \log(1/h)}\right)$ on every compact subset of $\mathbb{R}^n$, we a.s have that

$$||X(t) - X(s)|| \leq C\sqrt{|t-s|^\gamma\,(-1\log|t-s|)}.$$

For $0 < x < 1$, $-\log(x) \leq C_\varepsilon x^{-\varepsilon}$, and since $0 \leq |t-s| \leq 1$, we have a.s

$$||X(t) - X(s)|| \leq c(\varepsilon)\sqrt{|t-s|^\gamma(-|t-s|^{-\varepsilon})} \leq C(\varepsilon)|t-s|^{\gamma/2-\varepsilon},$$

where $C(\varepsilon) > 0$ is a constant. Hence $X \in \Lambda^{\frac{\gamma}{2}-\varepsilon}(E, \mathbb{R}^d)$ for every $\gamma/2 > \varepsilon > 0$. By Lemma 2.9 we get that

$$\dim_{\mathrm{H}} X(E) \leq \frac{2}{\gamma}\dim_{\mathrm{H}} E = \frac{1}{\beta}\dim_{\mathrm{H}} E. \tag{116}$$

Next, suppose that $\mathcal{H}^{\mathrm{h}}(E) > 0$. Using Frostman's lemma in form of Lemma 4.11 we find a measure $\nu \in \mathcal{M}^1(E)$ such that $\nu(B) \leq h(\operatorname{diam} B)$ for all balls

$B \subset \mathbb{R}^n$. We define measure μ as a push-forward $\mu = X_*\nu$ with the Fourier transform

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-i2\pi\xi\cdot X(u)} d\nu(u), \ \xi \in \mathbb{R}^d.$$

For integer $q \geq 1$ we would like to estimate $\mathbb{E}(|\hat{\mu}(\xi)|^{2q})$ as in the proof of Theorem 4.9, so once again we write it like in the proof of Theorem 4.8. For $t = (t_1, \ldots, t_q)$ and $s = (s_1, \ldots, s_q)$ write

$$\psi(t,s) = (2\pi)^2 d^{-1}\mathbb{E}\left(|X(t_1) + \ldots + X(t_q) - X(s_1) - \ldots - X(s_q)|^2\right).$$

Thus we obtain

$$\mathbb{E}(|\hat{\mu}(\xi)|^{2q}) = \int \cdots \int_{\mathbb{R}^n} e^{-\frac{1}{2}|\xi|^2\psi(t,s)} d\nu(t_1) d \cdots d\nu(t_q) d\nu(s_1) \cdots d\nu(s_q). \quad (117)$$

Next, we shall look for a lower bound for the function ψ. Taking $\psi(t,s)$ to a complex Hilbert space $\mathscr{H}$ with a linear isometry, $X_t \longleftrightarrow Y_t$ defined on (35), we obtain

$$\begin{aligned} \psi(t,s) &= \frac{1}{2}\int_{\mathbb{R}^n} |X(t_1) + \ldots + X(t_q) - X(s_1) - \ldots - X(s_q)|^2 c|x|^{-n-\gamma} dx \\ &= \frac{c}{2}\int_{\mathbb{R}^n} |e^{ixt_1} + \ldots + e^{ixt_q} - e^{ixs_1} - \ldots - e^{ixs_q}|^2 |x|^{-n-\gamma} dx. \end{aligned}$$

Let $\varepsilon > 0$ and suppose that $s = (s_1, \ldots, s_q)$ is given. We define two sets; First, let

$$F(s,\varepsilon) = \left\{ t \in \mathbb{R}^n : \inf_{i \leq j \leq q} |t - s_j| \leq \varepsilon \right\},$$

and then let

$$G(s,\varepsilon) = \{t = (t_1, \ldots, t_q) : t_k \in F(s,\varepsilon) \ \forall \ 1 \leq k \leq q\}.$$

The set $F(s,\varepsilon)$ can be covered with q balls of radius ε, so

$$\nu(F(s,\varepsilon)) \leq qh(2\varepsilon),$$

and therefore

$$\int \cdots \int_{G(s,\varepsilon)} 1 d\nu(t_1) \cdots d\nu(t_q) \leq (qh(2\varepsilon))^q. \quad (118)$$

Next, let us prove the following statement: If $t \notin G(s,\varepsilon)$, then for some constant $c_1 > 0$

$$\psi(t,s) > c_1\varepsilon^\gamma; \quad (119)$$

Suppose that $t \notin G(s,\varepsilon)$. We use a function with the following properties: Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be a function supported by the unit ball satisfying $0 \le \phi(x) \le 1$, $\phi(0) = 1$, $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$, and

$$\phi(x) = \int_{\mathbb{R}^n} e^{ix\cdot u}\varphi(u)du, \quad \phi_\varepsilon(x) = \int_{\mathbb{R}^n} e^{ix\cdot u}\varphi(\varepsilon u)du,$$

where the function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is the Fourier transform of the function ϕ to a constant. Now, for values $1 \le k \le q$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} (e^{iut_1} + \ldots + e^{iut_q} - e^{ius_1} - \ldots - e^{ius_q})e^{-iut_k}\varphi(\varepsilon u)du \\
&= \phi_\varepsilon(t_1 - t_k) + \ldots + \phi_\varepsilon(t_q - t_k) - \phi_\varepsilon(s_1 - t_k) - \ldots - \phi_\varepsilon(s_q - t_k) \ge \varepsilon^{-n}.
\end{aligned} \tag{120}$$

This is since $\phi_\varepsilon(t_k - t_k) = \varepsilon^{-n}$ and by assumption, for some $1 \le j \le q$, $|s_j - t_k| > \varepsilon$, so $\phi_\varepsilon(s_j - t_k) = 0$. Suppose otherwise; then $|s_j - t_k| \le \varepsilon$ for every $1 \le j \le q$ and $t \in G(s,\varepsilon)$, which is a contradiction. Therefore, applying Cauchy-Schwarz inequality to (120) we get

$$\begin{aligned}
\varepsilon^{-2n} &\le 2c^{-1}\psi(t,s)\int_{\mathbb{R}^n} |u|^{n+\gamma}|\varphi(\varepsilon u)|^2du \\
&= 2c^{-1}\psi(t,s)\int_{\mathbb{R}^n} \varepsilon^{-2n-\gamma}|u|^{n+\gamma}|\varphi(u)|^2du \\
&= c_1^{-1}\varepsilon^{-2n-\gamma}\psi(t,s),
\end{aligned}$$

where $c_1 = c_1(n,\gamma) > 0$, proving the statement (119). To continue estimating (117), for a given $\xi \in \mathbb{R}^d$ we choose $\varepsilon = |\xi|^{-2/\gamma}$. Writing

$$\begin{aligned}
\int\cdots\int e^{-\frac{1}{2}|\xi|^2\psi(t,s)}d\nu(t_1)\cdots d\nu(t_q) &= \int\cdots\int_{G(s,\varepsilon)} e^{-\frac{1}{2}|\xi|^2\psi(t,s)}d\nu(t_1)\cdots d\nu(t_q) \\
&+ \sum_{v=1}^{\infty}\int\cdots\int_{G(s,\varepsilon 2^v)\setminus G(s,\varepsilon 2^{v-1})} e^{-\frac{1}{2}|\xi|^2\psi(t,s)}d\nu(t_1)\cdots d\nu(t_q) = I_1 + I_2.
\end{aligned}$$

By using estimates (118) in I_1 and (119) in I_2 we get that

$$I_1 + I_2 \le (qh(2\varepsilon))^q + \sum_{v=1}^{\infty}\left(e^{-\frac{1}{2}c_1 2^{(v-1)\gamma}}(qh(2^{v+1}\varepsilon))^q\right) \le (Cqh(|\xi|^{-2/\gamma}))^q, \tag{121}$$

where the last inequality is due to assumption that $h(2\varepsilon) = \mathcal{O}(h(\varepsilon))$ as $\varepsilon \to 0$. On the other hand, for integrals with respect to $d\nu(s_1)\cdots d\nu(s_q)$ we can use the estimate

$$\int \cdots \int_{\mathbb{R}^n} e^{-\frac{1}{2}|u|^2\psi(t,s)} d\nu(s_1)\cdots d\nu(s_q) \leq (\nu(E))^q. \tag{122}$$

Thus combining (121) with (122) we get

$$\mathbb{E}\left(|\hat{\mu}(\xi)|^{2q}\right) \leq \left(C\nu(E)qh(|\xi|^{-2/\gamma})\right)^q = \left(C\nu(E)qh(|\xi|^{-1/\beta})\right)^q.$$

Then by Lemma 4.10 we have that a.s

$$\hat{\mu}(\xi) = \mathcal{O}\left(\sqrt{\log(|\xi|)h(|\xi|^{-1/\beta})}\right)$$

by same arguments as in the proof of Theorem 4.9.
To finish the proof, suppose $\dim_{\mathrm{H}} E = \alpha < n\beta$. Choosing, for example, concave function $h(t) = \log(1 + t^\alpha)$ and estimating logarithm like before, we get by Frostman's lemma that $X(E)$ carries a.s a measure $\mu \not\equiv 0$ whose Fourier transform satisfies

$$\hat{\mu}(\xi) = \mathcal{O}\left(|\xi|^{-\frac{\alpha}{2\beta}+\varepsilon}\right)$$

for every $\varepsilon > 0$. Therefore, if $\dim_{\mathrm{H}} E = \alpha < n\beta$, we have a.s

$$\dim_{\mathrm{F}} X(E) \geq \frac{\alpha}{\beta} \geq \dim_{\mathrm{H}} X(E),$$

and thus by (116) we have that $X(E)$ is a.s a Salem set of dimension α/β. $\square$

Like with Theorem 4.13, with Theorem 4.15 we get a Salem set of any dimension. Now however, we can also map from $\mathbb{R}^n$ to $\mathbb{R}^d$.

4.2.4 Sets under random diffeomorphism

In Section 3.2 we saw that there are large amounts of "nice" self-similar sets that are not Salem sets. On the other hand, we have now seen that there are plenty of random functions that almost surely map given sets into random Salem sets. Well then, is it possible to take a such "nice" set and perturb it a little bit to make it a Salem set? The answer to the question is positive; at least sort of. We are going to prove that every Borel set on $\mathbb{R}$ is diffeomorphic to a Salem set on $\mathbb{R}$.

Theorem 4.16. *For every Borel set $F \subset \mathbb{R}$ there exists a diffeomorphism $f : \mathbb{R} \to \mathbb{R}$ such that a.s*

$$\dim_{\mathrm{F}} f(F) \geq \dim_{\mathrm{H}} F.$$

The statement was proved in [2] and we base this section on it. This theorem is proof of existence and sadly it does not give us any concrete function to play with different sets. Additionally, at the time of writing, no concrete example of such diffeomorphism could be found. More information about possible further consequences and limitations of Theorem 4.16 can be found in [3].
Formally, a diffeomorphism is a differentiable bijective mapping from a set to another with a differentiable inverse mapping. Two sets are said to be diffeomorphic if there exists a diffeomorphism between them. Before we begin the proof of Theorem 4.16, let us talk briefly about what we are about to do. First, we need to fix a set, define our mapping and show that it is indeed a diffeomorphism. Then we check what happens to the Fourier dimension of the set if we map it. Given an arbitrary Borel set from $\mathbb{R}$ we consider what happens to the Fourier dimension if we map the intersection of the Borel set and the set we fixed. Like before, the proof will be divided into smaller parts. Next, we introduce some notation.
Let $E \subset \mathbb{R}$ be a compact set and let D be the set of all bounded connected components of the complement, $D \subset E^c$. In other words, we take the holes of the set E. For each component, $v \in D$ choose a non-negative number δ_v such that

$$\sum_{v \in D} \delta_v < \infty.$$

Then write

$$\Omega = \prod_{v \in D} [0, \delta_v] = [0, \delta_{v_1}] \times [0, \delta_{v_2}] \times \ldots,$$

and for $\omega \in \Omega$, $x \in E$, define a mapping

$$f_\omega(x) = x + \sum_{v \subset]-\infty, x[} \omega_v,$$

where $\omega_v \in [0, \delta_v]$ is random and the sum is taken over those v that lie "left" of the point x. This mapping widens each of the holes v of the set E by ω_v.
Next, let $\nu \in \mathcal{M}^1([0,1])$ be such that

$$\lim_{|\xi| \to \infty} \hat{\nu}(\xi) = 0,$$

let $\Delta_v(x) = \delta_v x$ and define a push-forward measure $\nu_v = \Delta_{v*}\nu$. Let $\mathcal{P}$ be a product measure on Ω such that its projection to the v-coordinate of $\omega \in \Omega$ is ν_v. We continue by extending the mapping f_ω from the set E to $\mathbb{R}$ by setting

$$f_\omega(x) = x + \sum_{v \in D} \omega_v \psi\left(\frac{x - \inf v}{|v|}\right),$$

where $\psi \in C^\infty$ is increasing such that

$$\psi(x) = \begin{cases} 0, & x \in]-\infty, 0] \\ 1, & x \in [1, \infty[\end{cases}$$

and ω_v is like before. Last, for positive integer m and for $0 \leq \alpha \leq 1$ we choose the values $\delta_v = |v|^m \delta(|v|)$, where

$$\delta(t) = \begin{cases} 1/\max\{-\log t, \log 2\}, & \text{if } \alpha = 0 \\ t^\alpha, & \text{if } \alpha \neq 0. \end{cases} \tag{123}$$

Definition 4.17. *On $\mathbb{R}$, we say that a family F of functions is uniformly equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $|x-y| < \delta$ and for every $f \in F$,*

$$|f(x) - f(y)| < \varepsilon.$$

Note, that δ may only depend on ε [13].

We may begin by checking that the function defined as above is a diffeomorphism. Let us give the following theorem.

Theorem 4.18. *Function f_ω is a $C^{m+\alpha}$-diffeomorphism for every $\omega \in \Omega$ and $\{f_\omega^{(m)}\}_{\omega \in \Omega}$ is uniformly equicontinuous with respect to modulus $2||\psi^{(m+1)}||_\infty \delta$.*

Before the proof of Theorem 4.18, for the rest of this section we need to consider the following lemma.

Lemma 4.19. *Let $\{I_k\}_{k=1}^\infty$ be a disjoint family of open intervals such that $I = \bigcup_k I_k$ is bounded. Let $\{g_k\}_{k=1}^\infty$ be a family of increasing $m \geq 1$ times differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ such that $\{g_k^{(m)}\}_{k=1}^\infty$ is uniformly equicontinuous with modulus ω and*

$$g_k(\inf I_k) = 0,\ g_k'(x) = \ldots = g_k^{(m)}(x) = 0 \text{ for } x \in I_k^c \text{ for all } k. \tag{124}$$

Define a function $g : \mathbb{R} \to \mathbb{R}$ by setting

$$g(x) = \sum_{k=1}^\infty g_k(x).$$

Then g is m-times differentiable function, $g(\inf I) = 0$, $g'(x) = \ldots = g^{(m)}(x) = 0$ for $x \in I^c$ and g^m is uniformly continuous with modulus 2ω.

The following proof is adapted from [2, p.110, Lemma 7].

Proof. Since $\{g_k^{(m)}\}_{k=1}^{\infty}$ is uniformly equicontinuous, by (124) for $x \in \mathbb{R}$

$$|g_k^{(m)}(x) - g_k^{(m)}(I_k^c)| = |g_k^{(m)}(x)| \le \omega(\mathrm{dist}(x, I_k^c)). \tag{125}$$

Also, for $t \ge 0$

$$g_k^{(m)}(\inf I_k + t) \le \omega(\mathrm{dist}(\inf I_k + t, I_k^c)) = \omega(t).$$

Thus $g_k^{(m-1)}(\inf I_k + t) \le \omega(t)t$, $g_k^{(m-2)}(\inf I_k + t) \le \frac{1}{1\cdot 2}\omega(t)t^2, \dots$ and

$$g_k(\inf I_k + t) \le \frac{\omega(t)}{m!} t^m \tag{126}$$

for all k. We have that

$$g(x) \le \sum_k |g_k(I_k)| \le \frac{\omega(\sup_k |I_k|)}{m!} \sum_k |I_k|^m < \infty,$$

so g is well defined. Additionally $g^{(m)}$ exists and is continuous on I as $g_k^{(m)}$ are uniformly continuous. By (124) for $x \in I$

$$\lim_{x \to I^c} g^{(m)}(x) = 0.$$

Next, to show that g is m-times continuously differentiable, we check that $g^{(m)}$ exists on I^c and is equal to 0. For this it is enough to take limit from the right side; mapping $x \to -g(-x) + |g(I)|$ gives the same form as g if we change g_k with $x \to -g_k(-x) + |g_k(I)|$. Fix a point $x \in I^c$ and let $h \ge 0$. Then we can write

$$g(x+h) - g(h) = \sum_{I_k \subset]x, x+h[} |g_k(I_k)| + \sum_{x+h \in I_k} g(x+h) = S_1 + S_2.$$

By (126), estimating

$$S_1 \le \frac{\omega(h)}{m!} \sum_{I_k \subset]x, x+h[} |I_k|^m \le \frac{\omega(h)}{m!} \left(\sum_{I_k \subset]x, x+h[} |I_k| \right)^m \le \frac{\omega(h)}{m!} h^m$$

and again, by (126),

$$S_2 = g(x+h) \le \frac{\omega(h)}{m!} h^m.$$

Thus for $h \ge 0$

$$g(x+h) - g(h) \le \frac{2\omega(h)}{m!} h^m,$$

so g is m-times differentiable with $g'(x) = \ldots = g^{(m)}(x) = 0$ for $x \in I^c$. Last, we show that $g^{(m)}$ is uniformly continuous with modulus 2ω : If for some $n \in \mathbb{N}$, $x, y \in \overline{I}_n$, then

$$|g^{(m)}(x) - g^{(m)}(y)| = |g_n^{(m)}(x) - g_n^{(m)}(y)| \leq \omega(|x - y|).$$

Otherwise there is an open interval on $]x, y[$ intersecting I^c and

$$\begin{aligned} |g^{(m)}(x) - g^{(m)}(y)| \leq |g^{(m)}(x)| + |g^{(m)}(y)| &\leq \omega(\mathrm{dist}(x, I^c)) + \omega(\mathrm{dist}(y, I^c)) \\ &\leq 2\omega(|x - y|), \end{aligned}$$

which concludes the proof. □

Now we can proof Theorem 4.18, adapted from [2, p.106, Theorem 3].

Proof. Let $g_v(x) = \omega_v \psi(\frac{x - \inf v}{|v|})$. Then for every k and $x \in \overline{v}$

$$g_v^k(x) = \frac{\omega_v}{|v|^k} \psi^{(k)} \left(\frac{x - \inf v}{|v|} \right).$$

Since $\delta(t)/t$ is decreasing, for $x, y \in \overline{v}$

$$\begin{aligned} |g_v^{(m)}(x) - g_v^{(m)}(y)| &\leq ||\psi^{(m+1)}||_\infty \frac{\delta_v}{|v|^{m+1}} |x - y| = ||\psi^{(m+1)}||_\infty \frac{\delta(|v|)}{|v|} |x - y| \\ &\leq ||\psi^{(m+1)}||_\infty \frac{\delta(|x - y|)}{|x - y|} |x - y| = ||\psi^{m+1}||_\infty \delta(|x - y|). \end{aligned}$$

Function $g_v^{(m)}$ is constant on v^c, so for any $x, y \in \mathbb{R}$

$$|g_v^{(m)}(x) - g_v^{(m)}(y)| \leq ||\psi^{m+1}||_\infty \delta(|x - y|).$$

Therefore the claim follows from Lemma 4.19, since

$$f_\omega(x) = x + \sum_{v \in D} g_v(x).$$

□

Let us use the following notation: If $J \subset \mathbb{R}$ is an interval and $x > 0$, then

$$\phi(J, x) = \#\{v \in D : v \subset J \text{ and } \delta_v \geq x^{-1}\}.$$

Next, under some assumptions, we will find a lower bound for the Fourier dimension of our push-forward measure, implying the lower bound for the Fourier dimension of the image of set E.

Theorem 4.20. *Let $0 \leq s \leq 1$ and let μ be a probability measure on set E. Suppose that there are constants a and $b > 0$ such that*

$$\phi(J, x) \geq a + b\,[\log \mu(J) + s \log x]$$

for every interval J and for $x \geq x_0$. Then a.s $\dim_{\mathrm{F}} f_{\omega*}\mu \geq s$.

The following proof is adapted from [2, p.107 Lemma 6, p.105 Theorem 2].

Proof. Let $(\Omega, \mathcal{P})$ be a probability space and $\omega \in \Omega$. Let $\omega \to \mu_\omega$ be a random probability measure on $\mathbb{R}$ such that a.s $\operatorname{diam}(\operatorname{spt} \mu_\omega) < M$, where $M > 0$ is a constant. Let us begin by proving the following statement: Suppose that for an integer $q \geq 1$

$$\mathbb{E}(|\hat{\mu}_\omega(\xi)|^{2q}) = \mathcal{O}(|\xi|^{-sq+1}). \tag{127}$$

Then a.s $\hat{\mu}_\omega(\xi) = \mathcal{O}(|\xi|^{-s/2+\varepsilon})$ for every $\varepsilon > 0$; By Fubini theorem, from (127) it follows that

$$\int \sum_{\xi \in \mathbb{Z}/M} |\xi|^{sq-3} |\hat{\mu}_\omega(\xi)|^{2q} d\mathcal{P}(\omega) = \sum_{\xi \in \mathbb{Z}/M} |\xi|^{sq-3} \mathbb{E}\left(|\hat{\mu}_\omega(\xi)|^{2q}\right)$$
$$\leq c \sum_{\xi \in \mathbb{Z}/M} |\xi|^{-2} < \infty,$$

where $c > 0$ is a constant and $\mathbb{Z}/M$ denotes the quotient space, or in other words $\mathbb{Z} \mod M$. Then for almost every $\omega \in \Omega$

$$\sum_{\xi \in \mathbb{Z}/M} |\xi|^{sq-3} |\hat{\mu}_\omega(\xi)|^{2q} < \infty.$$

Thus a.s for $\xi \in \mathbb{Z}/M$ we have

$$\lim_{|\xi| \to \infty} |\xi|^{sq-3} |\hat{\mu}_\omega(\xi)|^{2q} = 0,$$

and therefore a.s for $\xi \in \mathbb{Z}/M$

$$\hat{\mu}_\omega(\xi) = \mathcal{O}(|\xi|^{-s/2+3/2q}). \tag{128}$$

By Lemma 4.10 we get that (128) holds for every $\xi \in \mathbb{R}$ a.s. Letting $q \to \infty$ we get the claim for every $\varepsilon > 0$. Now, if we choose the random probability measure $\mu_\omega = f_{\omega*}\mu$, it is enough to show that for some integer $q \geq 1$

$$\mathbb{E}(|\hat{\mu}_\omega(\xi)|^{2q}) = \mathcal{O}(|\xi|^{-sq+1}).$$

Let us estimate $\mathbb{E}(|\hat{\mu}_\omega(\xi)|^{2q})$ like in the proof of Theorem 4.9. First, for $s = (s_1, \dots, s_q), t = (t_1, \dots, t_q)$ we get

$$|\hat{\mu}_\omega(\xi)|^{2q} = \int \cdots \int_{\mathbb{R}} e^{i2\pi\xi \sum_{k=1}^q [f_\omega(s_k) - f\omega(t_k)]} d\mu(t_1) \cdots d\mu(t_q) d\mu(s_1) \cdots d\mu(s_q).$$

Let us write $\eta(t, s) = \sum_{k=1}^q (s_k - t_k)$ and

$$\theta_{t,s}(\xi) = \#\{k : \xi < s_k\} - \#\{k : \xi < t_k\}.$$

Then, if $(t, s) \in E^{2q}$, $\theta_{t,s}(\xi)$ is a constant for each $v \in D$ and

$$\sum_{k=1}^q [f_\omega(s_k) - f\omega(t_k)] = \eta(t, s) + \sum_{v \in D} \theta_{t,s}(v)\omega_v.$$

By Fubini theorem and using the above notation we have

$$\begin{aligned}
&\mathbb{E}(|\hat{\mu}_\omega(\xi)|^{2q}) \\
&= \mathbb{E}\left(\int e^{i2\pi\xi\eta(t,s)} e^{i2\pi\xi \sum_{v \in D} \theta_{t,s}(v)\omega_v} d\mu(t_1) \cdots d\mu(t_q) d\mu(s_1) \cdots d\mu(s_q)\right) \\
&\leq \int \left| \int \prod_{v \in D} e^{i2\pi\xi\theta_{t,s}(v)\omega_v} d\mathcal{P}(\omega) \right| d\mu(t_1) \cdots d\mu(t_q) d\mu(s_1) \cdots d\mu(s_q) \\
&= \int \prod_{v \in D} \left| \int e^{i2\pi\xi\theta_{t,s}(v)\omega_v} d\nu_v(\omega_v) \right| d\mu(t_1) \cdots d\mu(t_q) d\mu(s_1) \cdots d\mu(s_q) \\
&= \int \prod_{v \in D} |\hat{\nu}_v(\xi\theta_{t,s}\delta_v)| \, d\mu(t_1) \cdots d\mu(t_q) d\mu(s_1) \cdots d\mu(s_q). \qquad (129)
\end{aligned}$$

Let $B_r = \{(t, s) : \mu(J) \leq r, J \text{ is an interval with } \theta_{t,s} \neq 0, t, s \in J\}$. If $(t, s) \in B_r$, for each t_k there is a s_j such that $\mu(]t_k, s_j[) \leq r$: This is since either $t_k = s_j$ for some j or $\theta_{t,s}$ increases by 1 at point t_k. Then for every fixed s there is a set A with $\mu(A) = 2qr$ and $t_k \in A$ for each $k = 1, \dots, q$ whenever $(t, s) \in B_r$. Therefore

$$\begin{aligned}
\underbrace{\mu \times \ldots \times \mu}_{2q}(B_r) &= \int \ldots \int \underbrace{\mu \times \ldots \times \mu}_{q}(\{t : (t, s) \in B_r\}) d\mu(s_1) \cdots d\mu(s_q) \\
&\leq (2qr)^q.
\end{aligned}$$

On the other hand, if $(t, s) \in B_r$ there exists an interval J such that $\mu(J) \geq r$ and $\theta_{t,s} \neq 0$ on J. Hence for any $K > 0$, $|\xi| \geq Kx_0$

$$\begin{aligned}
\prod_{v \in D} |\hat{\nu}_v(\xi\theta_{t,s}\delta_v)| \leq \prod_{v \in D, v \subset J} \varphi(\delta_v|\xi|) &\leq \varphi(K)^{\phi(J, K^{-1}|\xi|)} \\
&\leq \varphi(K)^{a + b[\log r + s\log(K^{-1}|\xi|)]},
\end{aligned}$$

where $\varphi = \sup_{|\xi|\geq x} |\hat{\nu}(\xi)|$ and r is a positive constant. Combining the above estimates with (129) we get

$$\mathbb{E}(|\hat{\mu}_\omega(\xi)|^{2q}) \leq (2qr)^q + \varphi(K)^{a+b[\log r + s\log(K^{-1}|\xi|)]}.$$

Let us choose number r by setting

$$\log r = -s\log|\xi| \left(\frac{a}{bs\log|\xi|} + \left(1 - \frac{\log K}{\log|\xi|}\right)\right) \frac{b\log\varphi(K)}{b\log\varphi(K) - q},$$

so that $\varphi(K)^{a+b[\log r + s\log(K^{-1}|\xi|)]} = r^q$. Fixing K and letting $|\xi| \to \infty$ we get

$$\mathbb{E}(|\hat{\mu}_\omega(\xi)|^{2q}) = \mathcal{O}(|\xi|^{-sq\frac{b\log\varphi(K)}{b\log\varphi(K)-q}+\frac{1}{2}}).$$

Then for K big enough

$$\mathbb{E}\left(|\hat{\mu}_\omega(\xi)|^{2q}\right) = \mathcal{O}\left(|\xi|^{-sq+1}\right),$$

which proves the theorem. $\square$

Next, let us construct the fixed set C as follows: Let $(c_k)_{k=1}^{\infty}$ be an increasing sequence of positive numbers such that $c_k \to \frac{1}{2}$ as $k \to \infty$, $\prod_{k=1}^{\infty} 2c_k > 0$, and

$$\lim_{k\to\infty} \frac{\log(1-2c_k)}{k} = 0.$$

For example, take $c_k = \frac{1}{2} - \frac{1}{3k^2}$. Let $C_0 = [0,1]$ and for $k \geq 1$, C_k is obtained from C_{k-1} by removing from each connected component $I \subset C_{k-1}$ the open middle interval of length $(1-2c_k)|I|$. Then C_k contains 2^k intervals of length $\prod_{i=1}^{k} c_k$. Finally, let $C = \bigcap_{k=1}^{\infty} C_k$. Also,

$$\mathcal{L}(C) = \lim_{k\to\infty} 2^k \prod_{i=1}^{k} c_i = \lim_{k\to\infty} \prod_{i=1}^{k} 2c_i > 0,$$

so C is a fat Cantor set. From now on, choose the set $E = C$. As promised before, we move on to estimating the Fourier dimension of images of intersections.

Theorem 4.21. *Let $F \subset \mathbb{R}$ be a Borel set with $\mathcal{H}^s(F) > 0$. Then a.s there exists $t \in \mathbb{R}$ such that*

$$\dim_{\mathrm{F}} f_\omega(C \cap (F+t)) \geq \frac{s}{m+\alpha}.$$

The following proof is adapted from [2, p.114 Lemma 9, p.115 Lemma 10, p.107 Theorem 5].

Proof. Let us first show that for some $t \in \mathbb{R}$ the intersection $C \cap (F+t)$ carries a probability measure ν such that $\nu(I) \leq c|I|^s$ for some constant $c > 0$ and for every interval I: Since $\mathcal{H}^s(F) > 0$, by Frostman's lemma there is a measure $\mu \in \mathcal{M}^1(F)$ such that

$$\mu(I) \leq c_0 |I|^s. \tag{130}$$

Using, first, the translation invariance of Lebesgue measure and Fubini theorem, second, we get that

$$\begin{aligned} 0 < \mu(\mathbb{R})\mathcal{L}^1(C) &= \iint \chi_C(t) dt d\mu(x) = \iint \chi_C(t+x) dt d\mu(x) \\ &= \iint \chi_C(x+t) d\mu(x) dt = \int \mu_t(C) dt, \end{aligned}$$

where μ_t denotes the translation of μ by t. Thus for some $t \in \mathbb{R}$, we have $\mu_t(C) > 0$. Fix such t and let

$$\nu = \frac{\mu_{t|C}}{\mu_t(C)}.$$

Then ν is a probability measure on $C \cap (F+t)$ and by (130)

$$\nu(I) \leq \frac{c_0}{\mu_t(C)} |I|^s \tag{131}$$

for every interval I as wanted. Next, to apply Theorem 4.20, we would like to show that there are constants a and $b > 0$, and a function θ such that $\theta(x) \to 1$ as $x \to \infty$ and

$$\#\{v \in D : v \subset J, |v| \geq x^{-1}\} \geq a + b[\log \nu(J) + s\theta(x) \log x] \tag{132}$$

for intervals J and $x \geq (1-2c_1)^{-1}$. This can be done as follows: Fix an interval J and a point $x > 0$, and let n be that positive integer for which

$$(1 - 2c_{n+2}) \prod_{i=1}^{n+1} c_i < x^{-1} \leq (1 - 2cn + 1) \prod_{i=1}^{n} c_i. \tag{133}$$

Now, think of the removed parts in the construction of the middle third Cantor set for example; It follows from the way the set of holes D is constructed that if there are intervals $v, v' \in D$ such that $v \neq v'$ and $|v| = |v'|$, there is an interval $V \in D$ contained between v and v' and $|V| > |v|$. Hence there is the largest $v \in D$ which intersects with J. Then there is connected $J' \subset J \setminus v$ for which by equation (131)

$$\nu(J') \geq \frac{\nu(J)}{2} \text{ and } |J'| \geq \left(\frac{\nu(J)}{2c}\right)^{1/s}.$$

Choose

$$k = \left\lceil \frac{\log(2c) - \log \nu(J)}{s \log 2} \right\rceil, \tag{134}$$

so that $\prod_{i=1}^{k} c_i \leq 2^{-k} \leq |J'|$. Therefore J' intersects some connected component smaller than v of $[0,1] \setminus C_k$ and hence $v \subset [0,1] \setminus C_k$. Thus J' contains atleast one of the connected components of C_k and therefore atleast 2^{n-k} elements of D which are in J, size of which is $(1 - 2c_{n+1}) \prod_{i=1}^{n} c_i$. We get that

$$\begin{aligned}
&\#\{v \in D : v \subset J, |v| \geq x^{-1}\} \geq 2^{n-k} \geq (n-k)\log 2 \\
&\geq -\frac{s\log 2 - \log(2c)}{s \log 2} + \frac{\nu(J)}{s} - \frac{n \log 2 \log x}{\log(1 - 2c_{n+2}) + \sum_{i=1}^{n+1} \log c_i}.
\end{aligned}$$

This is since

$$k \leq \frac{\log(2c) - \log \nu(J)}{s \log 2} + 1 = \frac{s \log 2 + \log(2c)}{s} - \frac{\nu(J)}{s},$$

and $(1-2c_{n+2}) \prod_{i=1}^{n+1} c_i < x^{-1}$, $\log x > -(\log(1-2c_{n+2}) + \sum_{i=1}^{n+1} \log c_i)$ leading to

$$1 > -\frac{\log x}{\log(1 - 2c_{n+2}) + \sum_{i=1}^{n+1} \log c_i},$$

which proves the equation (132) since $n \to \infty$ as $x \to \infty$ with constants and function θ

$$a = -\frac{s \log 2 + \log(2c)}{s}, \; b = \frac{1}{s}, \; \theta(x) = -\frac{n \log x}{\log(1 - 2c_{n+2}) + \sum_{i=1}^{n+1} \log c_i}.$$

Now, let $\varepsilon > 0$. Since ν is a probability measure on C, by (132) there are constants a and $b > 0$, and x_0 such that for intervals J and $x \geq x_0$

$$\begin{aligned}
\phi(J, x) &= \#\{v \in D : v \subset J, |v|^m \delta(|v|) \geq x^{-1}\} \\
&\geq \#\{v \in D : v \subset J, |v|^{m+\alpha+\varepsilon} \geq x^{-1}\} \\
&= \#\{v \in D : v \subset J, |v| \geq x^{-1/(m+\alpha+\varepsilon)}\} \\
&\geq a + b\left[\log \nu(J) + \left(\frac{s}{m+\alpha+\varepsilon} - \varepsilon\right) \log x\right].
\end{aligned}$$

Then by Theorem 4.20 a.s

$$\dim_{\mathrm{F}} f_{\omega*}\nu \geq \frac{s}{m+\alpha+\varepsilon} - \varepsilon.$$

Letting $\varepsilon \to 0$ by some sequence we get that a.s

$$\dim_{\mathrm{F}} f_{\omega *}\nu \geq \frac{s}{m+\alpha}.$$

Hence $f_\omega(C \cap (F+t))$ supports a.s a measure whose Fourier dimension is atleast $s/(m+\alpha)$ proving the theorem. $\square$

We can now prove Theorem 4.16, adapted from [2, p.105 Theorem 1] combined with notes from [3].

Proof. Let $0 < s \leq \dim_{\mathrm{H}} F$. This can be assumed since if $s = 0$, the statement becomes trivial. Let $(s_k)_{k=1}^{\infty}$ be an increasing sequence of positive numbers converging to s. Then there are open intervals $\{J_k\}_{k=1}^{\infty}$ such that for every k

$$\mathcal{H}^{s_k}(F \cap J_k) > 0:$$

Let $I_1 = \mathbb{R}$. For $k \geq 2$, suppose that intervals I_k have been defined such that

$$\dim_{\mathrm{H}}(F \cap I_k) = s.$$

Now, by [14, Theorem 13] there exists a compact subset $F_k \subset (F \cap I_k)$ such that $0 < \mathcal{H}^{s_k}(F_k) < \infty$. Let $x_k \in \mathbb{R}$ be such that

$$\mathcal{H}^{s_k}\left(F_k \cap]-\infty, x_k[\right) = \mathcal{H}^{s_k}\left(F_k \cap]x_k, \infty[\right).$$

Then $I_k \setminus \{x_k\}$ is union of two disjoint intervals. Choose I_{k+1} from either of those such that $\dim_{\mathrm{H}}(F \cap I_{k+1}) = s$ and denote the remaining interval by J_k. Then $\mathcal{H}^{s_k}(F \cap J_k) \geq \mathcal{H}^{s_k}(F_k)/2$. By induction, we get the rest of the intervals $\{J_k\}_{k=1}^{\infty}$.
Since the set C was chosen to construction of function f_ω, by Theorem 4.21 for each k there are $t_k \in \mathbb{R}$ and $\omega_k \in \Omega$ such that a.s

$$\dim_{\mathrm{F}} f_{\omega_k}(C \cap (F \cap J_k + t_k)) \geq \frac{s_k}{m+\alpha}.$$

Let $a_k = \inf J_k \cap (C - t_k)$ and $b_k = \sup J_k \cap (C - t_k)$. Define a function g_k

$$g_k(x) = \begin{cases} (f_{\omega_k}(x+t_k) - x) - (f_{\omega_k}(a_k+t_k) - a_k), & x \in [a_k, b_k], \\ 0, & x \in [a_k, b_k]^c. \end{cases}$$

By Theorem 4.18 f_{ω_k} is m-times differentiable and $f_{\omega_k}^{(m)}$ is uniformly continuous modulus $2||\psi^{(m+1)}||_\infty \delta$. Also $f'_{\omega_k} = 1, f''_{\omega_k} = 0, \ldots, f_{\omega_k}^{(m)} = 0$ on the set C.

Then g_k is m-times differentiable and $g_k^{(m)}$ is uniformly continuous modulus $2||\psi^{(m+1)}||_\infty\delta$. By Lemma 4.19 function

$$g(x) = \sum_{k=1}^{\infty} g_k(x)$$

is m-times differentiable and $g^{(m)}$ is uniformly continuous modulus $4||\psi^{(m+1)}||_\infty\delta$. Let $f(x) = x + g(x)$. For $x \in]a_n, b_n[$

$$f(x) = c + f_{\omega_n}(x + t_n)$$

and furthermore we have a.s

$$\dim_{\mathrm{F}} f(F) \geq \sup_{n\in\mathbb{N}} \dim_{\mathrm{F}} f(F \cap]a_n, b_n[) \geq \sup_{n\in\mathbb{N}} \frac{s_n}{m+\alpha} = \frac{s}{m+\alpha}. \tag{135}$$

Recalling (123), we may choose $m = 1$ and $\alpha = 0$ to obtain a function f which by Theorem 4.18 is a C^1 -diffeomorphism, and for which we a.s have by (135)

$$\dim_{\mathrm{F}} f(F) \geq s,$$

hence completing the proof. □

Note that the statement of (135) becomes empty as m tends to infinity. Some limitations and notes on the sharpness of the inequality in Theorem 4.16 were given in [3].

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